The Coq Proof Assistant
Reference Manual

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Version 8.2pl2

The Coq Development Team

TypiCal Project (formerly LogiCal)

1This research was partly supported by IST working group “Types”
Introduction

This document is the Reference Manual of version 8.2pl2 of the COQ proof assistant. A companion volume, the COQ Tutorial, is provided for the beginners. It is advised to read the Tutorial first. A book [14] on practical uses of the COQ system was published in 2004 and is a good support for both the beginner and the advanced user.

The COQ system is designed to develop mathematical proofs, and especially to write formal specifications, programs and to verify that programs are correct with respect to their specification. It provides a specification language named GALLINA. Terms of GALLINA can represent programs as well as properties of these programs and proofs of these properties. Using the so-called Curry-Howard isomorphism, programs, properties and proofs are formalized in the same language called Calculus of Inductive Constructions, that is a λ-calculus with a rich type system. All logical judgments in COQ are typing judgments. The very heart of the Coq system is the type-checking algorithm that checks the correctness of proofs, in other words that checks that a program complies to its specification. COQ also provides an interactive proof assistant to build proofs using specific programs called tactics.

All services of the COQ proof assistant are accessible by interpretation of a command language called the vernacular.

COQ has an interactive mode in which commands are interpreted as the user types them in from the keyboard and a compiled mode where commands are processed from a file.

- The interactive mode may be used as a debugging mode in which the user can develop his theories and proofs step by step, backtracking if needed and so on. The interactive mode is run with the coqtop command from the operating system (which we shall assume to be some variety of UNIX in the rest of this document).

- The compiled mode acts as a proof checker taking a file containing a whole development in order to ensure its correctness. Moreover, COQ’s compiler provides an output file containing a compact representation of its input. The compiled mode is run with the coqc command from the operating system.

These two modes are documented in Chapter 13.

Other modes of interaction with COQ are possible: through an emacs shell window, an emacs generic user-interface for proof assistant (ProofGeneral [1]) or through a customized interface (PCoq [128]). These facilities are not documented here. There is also a COQ Integrated Development Environment described in Chapter 15.

How to read this book

This is a Reference Manual, not a User Manual, then it is not made for a continuous reading. However, it has some structure that is explained below.
Introduction

- The first part describes the specification language, Gallina. Chapters 1 and 2 describe the concrete syntax as well as the meaning of programs, theorems and proofs in the Calculus of Inductive Constructions. Chapter 3 describes the standard library of Coq. Chapter 4 is a mathematical description of the formalism. Chapter 5 describes the module system.

- The second part describes the proof engine. It is divided in five chapters. Chapter 6 presents all commands (we call them vernacular commands) that are not directly related to interactive proving: requests to the environment, complete or partial evaluation, loading and compiling files. How to start and stop proofs, do multiple proofs in parallel is explained in Chapter 7. In Chapter 8, all commands that realize one or more steps of the proof are presented: we call them tactics. The language to combine these tactics into complex proof strategies is given in Chapter 9. Examples of tactics are described in Chapter 10.

- The third part describes how to extend the syntax of Coq. It corresponds to the Chapter 12.

- In the fourth part more practical tools are documented. First in Chapter 13, the usage of coqc (batch mode) and coqtop (interactive mode) with their options is described. Then, in Chapter 14, various utilities that come with the Coq distribution are presented. Finally, Chapter 15 describes the Coq integrated development environment.

At the end of the document, after the global index, the user can find specific indexes for tactics, vernacular commands, and error messages.

List of additional documentation

This manual does not contain all the documentation the user may need about Coq. Various informations can be found in the following documents:

- **Tutorial** A companion volume to this reference manual, the Coq Tutorial, is aimed at gently introducing new users to developing proofs in Coq without assuming prior knowledge of type theory. In a second step, the user can read also the tutorial on recursive types (document RecTutorial.ps).

- **Addendum** The fifth part (the Addendum) of the Reference Manual is distributed as a separate document. It contains more detailed documentation and examples about some specific aspects of the system that may interest only certain users. It shares the indexes, the page numbers and the bibliography with the Reference Manual. If you see in one of the indexes a page number that is outside the Reference Manual, it refers to the Addendum.

- **Installation** A text file INSTALL that comes with the sources explains how to install Coq.

- **The Coq standard library** A commented version of sources of the Coq standard library (including only the specifications, the proofs are removed) is given in the additional document Library.ps.
Credits

**COQ** is a proof assistant for higher-order logic, allowing the development of computer programs consistent with their formal specification. It is the result of about ten years of research of the Coq project. We shall briefly survey here three main aspects: the *logical language* in which we write our axiomatizations and specifications, the *proof assistant* which allows the development of verified mathematical proofs, and the *program extractor* which synthesizes computer programs obeying their formal specifications, written as logical assertions in the language.

The logical language used by **COQ** is a variety of type theory, called the *Calculus of Inductive Constructions*. Without going back to Leibniz and Boole, we can date the creation of what is now called mathematical logic to the work of Frege and Peano at the turn of the century. The discovery of antinomies in the free use of predicates or comprehension principles prompted Russell to restrict predicate calculus with a stratification of *types*. This effort culminated with *Principia Mathematica*, the first systematic attempt at a formal foundation of mathematics. A simplification of this system along the lines of simply typed \(\lambda\)-calculus occurred with Church’s *Simple Theory of Types*. The \(\lambda\)-calculus notation, originally used for expressing functionality, could also be used as an encoding of natural deduction proofs. This Curry-Howard isomorphism was used by N. de Bruijn in the Automath project, the first full-scale attempt to develop and mechanically verify mathematical proofs. This effort culminated with Jutting’s verification of Landau’s *Grundlagen* in the 1970’s. Exploiting this Curry-Howard isomorphism, notable achievements in proof theory saw the emergence of two type-theoretic frameworks; the first one, Martin-Löf’s *Intuitionistic Theory of Types*, attempts a new foundation of mathematics on constructive principles. The second one, Girard’s polymorphic \(\lambda\)-calculus \(\text{F}_\omega\), is a very strong functional system in which we may represent higher-order logic proof structures. Combining both systems in a higher-order extension of the Automath languages, T. Coquand presented in 1985 the first version of the *Calculus of Constructions*, CoC. This strong logical system allowed powerful axiomatizations, but direct inductive definitions were not possible, and inductive notions had to be defined indirectly through functional encodings, which introduced inefficiencies and awkwardness. The formalism was extended in 1989 by T. Coquand and C. Paulin with primitive inductive definitions, leading to the current *Calculus of Inductive Constructions*. This extended formalism is not rigorously defined here. Rather, numerous concrete examples are discussed. We refer the interested reader to relevant research papers for more information about the formalism, its meta-theoretic properties, and semantics. However, it should not be necessary to understand this theoretical material in order to write specifications. It is possible to understand the Calculus of Inductive Constructions at a higher level, as a mixture of predicate calculus, inductive predicate definitions presented as typed PROLOG, and recursive function definitions close to the language ML.

Automated theorem-proving was pioneered in the 1960’s by Davis and Putnam in propositional calculus. A complete mechanization (in the sense of a semi-decision procedure) of classical first-order logic was proposed in 1965 by J.A. Robinson, with a single uniform inference rule called *resolution*. Resolution relies on solving equations in free algebras (i.e. term structures), using the *unification algorithm*. 
Many refinements of resolution were studied in the 1970’s, but few convincing implementations were realized, except of course that PROLOG is in some sense issued from this effort. A less ambitious approach to proof development is computer-aided proof-checking. The most notable proof-checkers developed in the 1970’s were LCF, designed by R. Milner and his colleagues at U. Edinburgh, specialized in proving properties about denotational semantics recursion equations, and the Boyer and Moore theorem-prover, an automation of primitive recursion over inductive data types. While the Boyer-Moore theorem-prover attempted to synthesize proofs by a combination of automated methods, LCF constructed its proofs through the programming of tactics, written in a high-level functional meta-language, ML.

The salient feature which clearly distinguishes our proof assistant from say LCF or Boyer and Moore’s, is its possibility to extract programs from the constructive contents of proofs. This computational interpretation of proof objects, in the tradition of Bishop’s constructive mathematics, is based on a realizability interpretation, in the sense of Kleene, due to C. Paulin. The user must just mark his intention by separating in the logical statements the assertions stating the existence of a computational object from the logical assertions which specify its properties, but which may be considered as just comments in the corresponding program. Given this information, the system automatically extracts a functional term from a consistency proof of its specifications. This functional term may be in turn compiled into an actual computer program. This methodology of extracting programs from proofs is a revolutionary paradigm for software engineering. Program synthesis has long been a theme of research in artificial intelligence, pioneered by R. Waldinger. The Tablog system of Z. Manna and R. Waldinger allows the deductive synthesis of functional programs from proofs in tableau form of their specifications, written in a variety of first-order logic. Development of a systematic programming logic, based on extensions of Martin-Löf’s type theory, was undertaken at Cornell U. by the Nuprl team, headed by R. Constable. The first actual program extractor, PX, was designed and implemented around 1985 by S. Hayashi from Kyoto University. It allows the extraction of a LISP program from a proof in a logical system inspired by the logical formalisms of S. Feferman. Interest in this methodology is growing in the theoretical computer science community. We can foresee the day when actual computer systems used in applications will contain certified modules, automatically generated from a consistency proof of their formal specifications. We are however still far from being able to use this methodology in a smooth interaction with the standard tools from software engineering, i.e. compilers, linkers, run-time systems taking advantage of special hardware, debuggers, and the like. We hope that Coq can be of use to researchers interested in experimenting with this new methodology.

A first implementation of CoC was started in 1984 by G. Huet and T. Coquand. Its implementation language was CAML, a functional programming language from the ML family designed at INRIA in Rocquencourt. The core of this system was a proof-checker for CoC seen as a typed \( \lambda \)-calculus, called the Constructive Engine. This engine was operated through a high-level notation permitting the declaration of axioms and parameters, the definition of mathematical types and objects, and the explicit construction of proof objects encoded as \( \lambda \)-terms. A section mechanism, designed and implemented by G. Dowek, allowed hierarchical developments of mathematical theories. This high-level language was called the Mathematical Vernacular. Furthermore, an interactive Theorem Prover permitted the incremental construction of proof trees in a top-down manner, subgoaling recursively and backtracking from dead-alleys. The theorem prover executed tactics written in CAML, in the LCF fashion. A basic set of tactics was predefined, which the user could extend by his own specific tactics. This system (Version 4.10) was released in 1989. Then, the system was extended to deal with the new calculus with inductive types by C. Paulin, with corresponding new tactics for proofs by induction. A new standard set of tactics was streamlined, and the vernacular extended for tactics execution. A package to compile programs extracted from proofs to actual computer programs in CAML or some other functional language was designed and implemented by B. Werner. A new user-interface, relying on a CAML-X interface by D.
de Rauglaudre, was designed and implemented by A. Felty. It allowed operation of the theorem-prover through the manipulation of windows, menus, mouse-sensitive buttons, and other widgets. This system (Version 5.6) was released in 1991.

**Coq** was ported to the new implementation Caml-light of X. Leroy and D. Doligez by D. de Rauglaudre (Version 5.7) in 1992. A new version of Coq was then coordinated by C. Murthy, with new tools designed by C. Parent to prove properties of ML programs (this methodology is dual to program extraction) and a new user-interaction loop. This system (Version 5.8) was released in May 1993. A Centaur interface CTCOQ was then developed by Y. Bertot from the Croap project from INRIA-Sophia-Antipolis.

In parallel, G. Dowek and H. Herbelin developed a new proof engine, allowing the general manipulation of existential variables consistently with dependent types in an experimental version of Coq (V5.9).

The version V5.10 of Coq is based on a generic system for manipulating terms with binding operators due to Chet Murthy. A new proof engine allows the parallel development of partial proofs for independent subgoals. The structure of these proof trees is a mixed representation of derivation trees for the Calculus of Inductive Constructions with abstract syntax trees for the tactics scripts, allowing the navigation in a proof at various levels of details. The proof engine allows generic environment items managed in an object-oriented way. This new architecture, due to C. Murthy, supports several new facilities which make the system easier to extend and to scale up:

- User-programmable tactics are allowed
- It is possible to separately verify development modules, and to load their compiled images without verifying them again - a quick relocation process allows their fast loading
- A generic parsing scheme allows user-definable notations, with a symmetric table-driven pretty-printer
- Syntactic definitions allow convenient abbreviations
- A limited facility of meta-variables allows the automatic synthesis of certain type expressions, allowing generic notations for e.g. equality, pairing, and existential quantification.

In the Fall of 1994, C. Paulin-Mohring replaced the structure of inductively defined types and families by a new structure, allowing the mutually recursive definitions. P. Manoury implemented a translation of recursive definitions into the primitive recursive style imposed by the internal recursion operators, in the style of the ProPre system. C. Muñoz implemented a decision procedure for intuitionistic propositional logic, based on results of R. Dyckhoff. J.C. Filliâtre implemented a decision procedure for first-order logic without contraction, based on results of J. Ketonen and R. Weyhrauch. Finally C. Murthy implemented a library of inversion tactics, relieving the user from tedious definitions of “inversion predicates”.

Rocquencourt, Feb. 1st 1995
Gérard Huet

**Credits: addendum for version 6.1**

The present version 6.1 of Coq is based on the V5.10 architecture. It was ported to the new language Objective Caml by Bruno Barras. The underlying framework has slightly changed and allows more conversions between sorts.
The new version provides powerful tools for easier developments.

Cristina Cornes designed an extension of the COQ syntax to allow definition of terms using a powerful pattern-matching analysis in the style of ML programs.

Amokrane Saïbi wrote a mechanism to simulate inheritance between types families extending a proposal by Peter Aczel. He also developed a mechanism to automatically compute which arguments of a constant may be inferred by the system and consequently do not need to be explicitly written.

Yann Coscoy designed a command which explains a proof term using natural language. Pierre Crégut built a new tactic which solves problems in quantifier-free Presburger Arithmetic. Both functionalities have been integrated to the COQ system by Hugo Herbelin.

Samuel Boutin designed a tactic for simplification of commutative rings using a canonical set of rewriting rules and equality modulo associativity and commutativity.

Finally the organisation of the COQ distribution has been supervised by Jean-Christophe Filliâtre with the help of Judicaël Courant and Bruno Barras.

Lyon, Nov. 18th 1996
Christine Paulin

Credits: addendum for version 6.2

In version 6.2 of COQ, the parsing is done using camlp4, a preprocessor and pretty-printer for CAML designed by Daniel de Rauglaudre at INRIA. Daniel de Rauglaudre made the first adaptation of COQ for camlp4, this work was continued by Bruno Barras who also changed the structure of COQ abstract syntax trees and the primitives to manipulate them. The result of these changes is a faster parsing procedure with greatly improved syntax-error messages. The user-interface to introduce grammar or pretty-printing rules has also changed.

Eduardo Giménez redesigned the internal tactic libraries, giving uniform names to Caml functions corresponding to COQ tactic names.

Bruno Barras wrote new more efficient reductions functions.

Hugo Herbelin introduced more uniform notations in the COQ specification language: the definitions by fixpoints and pattern-matching have a more readable syntax. Patrick Loiseleur introduced user-friendly notations for arithmetic expressions.

New tactics were introduced: Eduardo Giménez improved a mechanism to introduce macros for tactics, and designed special tactics for (co)inductive definitions; Patrick Loiseleur designed a tactic to simplify polynomial expressions in an arbitrary commutative ring which generalizes the previous tactic implemented by Samuel Boutin. Jean-Christophe Filliâtre introduced a tactic for refining a goal, using a proof term with holes as a proof scheme.

David Delahaye designed the SearchIsos tool to search an object in the library given its type (up to isomorphism).

Henri Laulhière produced the COQ distribution for the Windows environment.

Finally, Hugo Herbelin was the main coordinator of the COQ documentation with principal contributions by Bruno Barras, David Delahaye, Jean-Christophe Filliâtre, Eduardo Giménez, Hugo Herbelin and Patrick Loiseleur.

Orsay, May 4th 1998
Christine Paulin
Credits: addendum for version 6.3

The main changes in version V6.3 was the introduction of a few new tactics and the extension of the
guard condition for fixpoint definitions.

B. Barras extended the unification algorithm to complete partial terms and solved various tricky bugs
related to universes.
D. Delahaye developed the AutoRewrite tactic. He also designed the new behavior of Intro and
provided the tacticals First and Solve.
J.-C. Filliâtre developed the Correctness tactic.
E. Giménez extended the guard condition in fixpoints.
H. Herbelin designed the new syntax for definitions and extended the Induction tactic.
P. Loiseleur developed the Quote tactic and the new design of the Auto tactic, he also introduced the
index of errors in the documentation.
C. Paulin wrote the Focus command and introduced the reduction functions in definitions, this last
feature was proposed by J.-F. Monin from CNET Lannion.

Orsay, Dec. 1999
Christine Paulin

Credits: versions 7

The version V7 is a new implementation started in September 1999 by Jean-Christophe Filliâtre. This
is a major revision with respect to the internal architecture of the system. The COQ version 7.0 was
distributed in March 2001, version 7.1 in September 2001, version 7.2 in January 2002, version 7.3 in

Jean-Christophe Filliâtre designed the architecture of the new system, he introduced a new repre-
sentation for environments and wrote a new kernel for type-checking terms. His approach was to use
functional data-structures in order to get more sharing, to prepare the addition of modules and also to
get closer to a certified kernel.

Hugo Herbelin introduced a new structure of terms with local definitions. He introduced “qualified”
names, wrote a new pattern-matching compilation algorithm and designed a more compact algorithm
for checking the logical consistency of universes. He contributed to the simplification of COQ internal
structures and the optimisation of the system. He added basic tactics for forward reasoning and coercions
in patterns.

David Delahaye introduced a new language for tactics. General tactics using pattern-matching on
goals and context can directly be written from the COQ toplevel. He also provided primitives for the
design of user-defined tactics in CAML.

Micaela Mayero contributed the library on real numbers. Olivier Desmettre extended this library
with axiomatic trigonometric functions, square, square roots, finite sums, Chasles property and basic
plane geometry.

Jean-Christophe Filliâtre and Pierre Letouzey redesigned a new extraction procedure from COQ
terms to CAML or HASKELL programs. This new extraction procedure, unlike the one implemented
in previous version of COQ is able to handle all terms in the Calculus of Inductive Constructions, even
involving universes and strong elimination. P. Letouzey adapted user contributions to extract ML pro-
grams when it was sensible. Jean-Christophe Filliâtre wrote coqdoc, a documentation tool for COQ
libraries usable from version 7.2.
Bruno Barras improved the reduction algorithms efficiency and the confidence level in the correctness of COQ critical type-checking algorithm.

Yves Bertot designed the SearchPattern and SearchRewrite tools and the support for the PCOQ interface (http://www-sop.inria.fr/lemme/pcq/).

Micaela Mayero and David Delahaye introduced Field, a decision tactic for commutative fields.

Christine Paulin changed the elimination rules for empty and singleton propositional inductive types.

Loïc Pottier developed Fourier, a tactic solving linear inequalities on real numbers.

Pierre Crégut developed a new version based on reflexion of the Omega decision tactic.

Claudio Sacerdoti Coen designed an XML output for the COQ modules to be used in the Hypertextual Electronic Library of Mathematics (HELM cf http://www.cs.unibo.it/helm).

A library for efficient representation of finite maps using binary trees contributed by Jean Goubault was integrated in the basic theories.

Pierre Courtieu developed a command and a tactic to reason on the inductive structure of recursively defined functions.

Jacek Chrząszcz designed and implemented the module system of COQ whose foundations are in Judicaël Courant’s PhD thesis.

The development was coordinated by C. Paulin.

Many discussions within the Démons team and the LogiCal project influenced significantly the design of COQ especially with J. Courant, J. Duprat, J. Goubault, A. Miquel, C. Marché, B. Monate and B. Werner.

Intensive users suggested improvements of the system: Y. Bertot, L. Pottier, L. Théry, P. Zimmerman from INRIA, C. Alvarado, P. Crégut, J.-F. Monin from France Telecom R & D.

Orsay, May. 2002
Hugo Herbelin & Christine Paulin

Credits: version 8.0

COQ version 8 is a major revision of the COQ proof assistant. First, the underlying logic is slightly different. The so-called impredicativity of the sort Set has been dropped. The main reason is that it is inconsistent with the principle of description which is quite a useful principle for formalizing mathematics within classical logic. Moreover, even in an constructive setting, the impredicativity of Set does not add so much in practice and is even subject of criticism from a large part of the intuitionistic mathematician community. Nevertheless, the impredicativity of Set remains optional for users interested in investigating mathematical developments which rely on it.

Secondly, the concrete syntax of terms has been completely revised. The main motivations were

- a more uniform, purified style: all constructions are now lowercase, with a functional programming perfume (e.g. abstraction is now written fun), and more directly accessible to the novice (e.g. dependent product is now written forall and allows omission of types). Also, parentheses and are no longer mandatory for function application.

- extensibility: some standard notations (e.g. “<” and “>”) were incompatible with the previous syntax. Now all standard arithmetic notations (=, +, *, /, <, <=, ... and more) are directly part of the syntax.
Together with the revision of the concrete syntax, a new mechanism of *interpretation scopes* permits to reuse the same symbols (typically +, -, *, /, <, <=) in various mathematical theories without any ambiguities for CoQ, leading to a largely improved readability of CoQ scripts. New commands to easily add new symbols are also provided.

Coming with the new syntax of terms, a slight reform of the tactic language and of the language of commands has been carried out. The purpose here is a better uniformity making the tactics and commands easier to use and to remember.

Thirdly, a restructuration and uniformisation of the standard library of CoQ has been performed. There is now just one Leibniz’ equality usable for all the different kinds of CoQ objects. Also, the set of real numbers now lies at the same level as the sets of natural and integer numbers. Finally, the names of the standard properties of numbers now follow a standard pattern and the symbolic notations for the standard definitions as well.

The fourth point is the release of CoQIDE, a new graphical gtk2-based interface fully integrated to CoQ. Close in style from the Proof General Emacs interface, it is faster and its integration with CoQ makes interactive developments more friendly. All mathematical Unicode symbols are usable within CoQIDE.

Finally, the module system of CoQ completes the picture of CoQ version 8.0. Though released with an experimental status in the previous version 7.4, it should be considered as a salient feature of the new version.

Besides, CoQ comes with its load of novelties and improvements: new or improved tactics (including a new tactic for solving first-order statements), new management commands, extended libraries.

Bruno Barras and Hugo Herbelin have been the main contributors of the reflexion and the implementation of the new syntax. The smart automatic translator from old to new syntax released with CoQ is also their work with contributions by Olivier Desmettre.

Hugo Herbelin is the main designer and implementor of the notion of interpretation scopes and of the commands for easily adding new notations.

Pierre Corbineau is the main designer and implementor of the new tactic for solving first-order statements in presence of inductive types. He is also the maintainer of the non-domain specific automation tactics.

Benjamin Monate is the developer of the CoQIDE graphical interface with contributions by Jean-Christophe Filliâtre, Pierre Letouzey, Claude Marché and Bruno Barras.


Pierre Letouzey and Jacek Chrząszcz respectively maintained the extraction tool and module system of CoQ.

Jean-Christophe Filliâtre, Pierre Letouzey, Hugo Herbelin and contributors from Sophia-Antipolis and Nijmegen participated to the extension of the library.

Julien Narboux built a NSIS-based automatic CoQ installation tool for the Windows platform.

Hugo Herbelin and Christine Paulin coordinated the development which was under the responsibility of Christine Paulin.

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Palaiseau & Orsay, Apr. 2004
Hugo Herbelin & Christine Paulin
(updated Apr. 2006)
Credits: version 8.1

COQ version 8.1 adds various new functionalities.

Benjamin Grégoire implemented an alternative algorithm to check the convertibility of terms in the COQ type-checker. This alternative algorithm works by compilation to an efficient bytecode that is interpreted in an abstract machine similar to Xavier Leroy’s ZINC machine. Convertibility is performed by comparing the normal forms. This alternative algorithm is specifically interesting for proofs by reflection. More generally, it is convenient in case of intensive computations.

Christine Paulin implemented an extension of inductive types allowing recursively non uniform parameters. Hugo Herbelin implemented sort-polymorphism for inductive types.

Claudio Sacerdoti Coen improved the tactics for rewriting on arbitrary compatible equivalence relations. He also generalized rewriting to arbitrary transition systems.

Claudio Sacerdoti Coen added new features to the module system.

Benjamin Grégoire, Assia Mahboubi and Bruno Barras developed a new more efficient and more general simplification algorithm on rings and semi-rings.

Laurent Théry and Bruno Barras developed a new significantly more efficient simplification algorithm on fields.

Hugo Herbelin, Pierre Letouzey, Julien Forest, Julien Narboux and Claudio Sacerdoti Coen added new tactic features.

Hugo Herbelin implemented matching on disjunctive patterns.

New mechanisms made easier the communication between COQ and external provers. Nicolas Ayache and Jean-Christophe Filliâtre implemented connections with the provers CVCL, SIMPLIFY and ZENON. Hugo Herbelin implemented an experimental protocol for calling external tools from the tactic language.

Matthieu Sozeau developed RUSSELL, an experimental language to specify the behavior of programs with subtypes.

A mechanism to automatically use some specific tactic to solve unresolved implicit has been implemented by Hugo Herbelin.

Laurent Théry’s contribution on strings and Pierre Letouzey and Jean-Christophe Filliâtre’s contribution on finite maps have been integrated to the COQ standard library. Pierre Letouzey developed a library about finite sets “à la Objective Caml”. With Jean-Marc Notin, he extended the library on lists. Pierre Letouzey’s contribution on rational numbers has been integrated and extended.

Pierre Corbineau extended his tactic for solving first-order statements. He wrote a reflection-based intuitionistic tautology solver.

Pierre Courtieu, Julien Forest and Yves Bertot added extra support to reason on the inductive structure of recursively defined functions.

Jean-Marc Notin significantly contributed to the general maintenance of the system. He also took care of coqdoc.

Pierre Castéran contributed to the documentation of (co-)inductive types and suggested improvements to the libraries.

Pierre Corbineau implemented the C-zar mathematical proof language, usable in combination with the tactic-based style of proof.

Finally, many users suggested improvements of the system through the Coq-Club mailing list and bug-tracker systems, especially user groups from INRIA Rocquencourt, Radbout University, University of Pennsylvania and Yale University.

Palaiseau, July 2006
Hugo Herbelin
Credits: version 8.2

COQ version 8.2 adds new features, new libraries and improves on many various aspects.

Regarding the language of COQ, the main novelty is the introduction by Matthieu Sozeau of a package of commands providing Haskell-style type classes. Type classes, that come with a few convenient features such as type-based resolution of implicit arguments, plays a new role of landmark in the architecture of COQ with respect to automatization. For instance, thanks to type classes support, Matthieu Sozeau could implement a new resolution-based version of the tactics dedicated to rewriting on arbitrary transitive relations.

Another major improvement of COQ 8.2 is the evolution of the arithmetics libraries and of the tools associated to them. Benjamin Grégoire and Laurent Théry contributed a modular library for building arbitrarily large integers from bounded integers while Evgeny Makarov contributed a modular library of abstract natural and integer arithmetics together with a few convenient tactics. On his side, Pierre Letouzey made numerous extensions to the arithmetics libraries on \( \mathbb{Z} \) and \( \mathbb{Q} \), including extra support for automatization in presence of various number-theory concepts.

Frédéric Besson contributed a reflexive tactic based on Krivine-Stengle Positivstellensatz (the easy way) for validating provability of systems of inequalities. The platform is flexible enough to support the validation of any algorithm able to produce a “certificate” for the Positivstellensatz and this covers the case of Fourier-Motzkin (for linear systems in \( \mathbb{Q} \) and \( \mathbb{R} \)), Fourier-Motzkin with cutting planes (for linear systems in \( \mathbb{Z} \)) and sum-of-squares (for non-linear systems). Evgeny Makarov made the platform generic over arbitrary ordered rings.

Arnaud Spiwack developed a library of 31-bits machine integers and, relying on Benjamin Grégoire and Laurent Théry’s library, delivered a library of unbounded integers in base \( 2^{31} \). As importantly, he developed a notion of “retro-knowledge” so as to safely extend the kernel-located bytecode-based efficient evaluation algorithm of COQ version 8.1 to use 31-bits machine arithmetics for efficiently computing with the library of integers he developed.

Beside the libraries, various improvements contributed to provide a more comfortable end-user language and more expressive tactic language. Hugo Herbelin and Matthieu Sozeau improved the pattern-matching compilation algorithm (detection of impossible clauses in pattern-matching, automatic inference of the return type). Hugo Herbelin, Pierre Letouzey and Matthieu Sozeau contributed various new convenient syntactic constructs and new tactics or tactic features: more inference of redundant information, better unification, better support for proof or definition by fixpoint, more expressive rewriting tactics, better support for meta-variables, more convenient notations, ...

Elie Soubiran improved the module system, adding new features (such as an “include” command) and making it more flexible and more general. He and Pierre Letouzey improved the support for modules in the extraction mechanism.

Matthieu Sozeau extended the RUSSELL language, ending in a convenient way to write programs of given specifications, Pierre Corbineau extended the C-zar mathematical proof language and the automatization tools that accompany it and added its documentation to the Reference Manual, Pierre Letouzey supervised and extended various parts the standard library, Stéphane Glondou contributed a few tactics and improvements, Jean-Marc Notin provided help in debugging, general maintenance and coqdoc support, Vincent Siles contributed extensions of the Scheme command and of injection.

Bruno Barras implemented the coqchk tool: this is a stand-alone type-checker that can be used to certify .vo files. Especially, as this verifier runs in a separate process, it is granted not to be “hijacked” by virtually malicious extensions added to COQ.

Yves Bertot, Jean-Christophe Filliâtre, Pierre Courtieu and Julien Forest acted as maintainers of features they implemented in previous versions of COQ.
Julien Narboux contributed to CoqIDE. Nicolas Tabareau made the adaptation of the interface of the old “setoid rewrite” tactic to the new version. Lionel Mamane worked on the interaction between Coq and its external interfaces. With Samuel Mimram, he also helped making Coq compatible with recent software tools. Russell O’Connor, Cezary Kaliszczyk, Milad Niqui contributed to improved the libraries of integers, rational, and real numbers. We also thank many users and partners for suggestions and feedback, in particular Pierre Castéran and Arthur Charguéraud, the INRIA Marelle team, Georges Gonthier and the INRIA-Microsoft Mathematical Components team, the Foundations group at Radbout university in Nijmegen, reporters of bugs and participants to the Coq-Club mailing list.

Palaiseau, June 2008
Hugo Herbelin
# Table of contents

## I The language  

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>The GALLINA specification language</td>
<td>31</td>
</tr>
<tr>
<td>1.1</td>
<td>Lexical conventions</td>
<td>31</td>
</tr>
<tr>
<td>1.2</td>
<td>Terms</td>
<td>33</td>
</tr>
<tr>
<td>1.2.1</td>
<td>Syntax of terms</td>
<td>33</td>
</tr>
<tr>
<td>1.2.2</td>
<td>Types</td>
<td>33</td>
</tr>
<tr>
<td>1.2.3</td>
<td>Qualified identifiers and simple identifiers</td>
<td>33</td>
</tr>
<tr>
<td>1.2.4</td>
<td>Numerals</td>
<td>33</td>
</tr>
<tr>
<td>1.2.5</td>
<td>Sorts</td>
<td>33</td>
</tr>
<tr>
<td>1.2.6</td>
<td>Binders</td>
<td>34</td>
</tr>
<tr>
<td>1.2.7</td>
<td>Abstractions</td>
<td>36</td>
</tr>
<tr>
<td>1.2.8</td>
<td>Products</td>
<td>36</td>
</tr>
<tr>
<td>1.2.9</td>
<td>Applications</td>
<td>36</td>
</tr>
<tr>
<td>1.2.10</td>
<td>Type cast</td>
<td>36</td>
</tr>
<tr>
<td>1.2.11</td>
<td>Inferable subterms</td>
<td>36</td>
</tr>
<tr>
<td>1.2.12</td>
<td>Local definitions (let-in)</td>
<td>36</td>
</tr>
<tr>
<td>1.2.13</td>
<td>Definition by case analysis</td>
<td>37</td>
</tr>
<tr>
<td>1.2.14</td>
<td>Recursive functions</td>
<td>38</td>
</tr>
<tr>
<td>1.3</td>
<td>The Vernacular</td>
<td>39</td>
</tr>
<tr>
<td>1.3.1</td>
<td>Declarations</td>
<td>40</td>
</tr>
<tr>
<td>1.3.2</td>
<td>Definitions</td>
<td>41</td>
</tr>
<tr>
<td>1.3.3</td>
<td>Inductive definitions</td>
<td>42</td>
</tr>
<tr>
<td>1.3.4</td>
<td>Definition of recursive functions</td>
<td>47</td>
</tr>
<tr>
<td>1.3.5</td>
<td>Statement and proofs</td>
<td>51</td>
</tr>
</tbody>
</table>

## 2 Extensions of GALLINA  

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>Record types</td>
<td>53</td>
</tr>
<tr>
<td>2.1</td>
<td>Variants and extensions of match</td>
<td>56</td>
</tr>
<tr>
<td>2.2.1</td>
<td>Multiple and nested pattern-matching</td>
<td>56</td>
</tr>
<tr>
<td>2.2.2</td>
<td>Pattern-matching on boolean values: the if expression</td>
<td>56</td>
</tr>
<tr>
<td>2.2.3</td>
<td>Irrefutable patterns: the destructuring let variants</td>
<td>57</td>
</tr>
<tr>
<td>2.2.4</td>
<td>Controlling pretty-printing of match expressions</td>
<td>58</td>
</tr>
<tr>
<td>2.3</td>
<td>Advanced recursive functions</td>
<td>60</td>
</tr>
<tr>
<td>2.4</td>
<td>Section mechanism</td>
<td>62</td>
</tr>
<tr>
<td>2.4.1</td>
<td>Section ident</td>
<td>63</td>
</tr>
<tr>
<td>2.4.2</td>
<td>End ident</td>
<td>63</td>
</tr>
</tbody>
</table>
# Table of contents

2.5 Module system .................................................. 63
  2.5.1 Module `ident` ........................................... 64
  2.5.2 End `ident` ............................................... 65
  2.5.3 Module `ident := module_expression` ..................... 65
  2.5.4 Module Type `ident` ..................................... 65
  2.5.5 End `ident` ............................................... 66
  2.5.6 Module Type `ident := module_type` ..................... 66
  2.5.7 Declare Module `ident : module_type` .................... 66
  2.5.8 Import `qualid` .......................................... 69
  2.5.9 Print Module `ident` ..................................... 70
  2.5.10 Print Module Type `ident` ................................ 70
  2.5.11 Locate Module `qualid` .................................. 70
2.6 Libraries and qualified names .................................. 70
  2.6.1 Names of libraries and files ................................ 70
  2.6.2 Qualified names .......................................... 71
2.7 Implicit arguments ............................................. 72
  2.7.1 The different kinds of implicit arguments ............... 72
  2.7.2 Maximal or non maximal insertion of implicit arguments .. 73
  2.7.3 Casual use of implicit arguments ........................ 73
  2.7.4 Declaration of implicit arguments for a constant .......... 73
  2.7.5 Automatic declaration of implicit arguments for a constant 76
  2.7.6 Mode for automatic declaration of implicit arguments .... 77
  2.7.7 Controlling strict implicit arguments .................... 77
  2.7.8 Controlling contextual implicit arguments ............... 78
  2.7.9 Controlling reversible-pattern implicit arguments ........ 78
  2.7.10 Controlling the insertion of implicit arguments not followed by explicit arguments ........ 78
  2.7.11 Explicit applications ................................... 78
  2.7.12 Displaying what the implicit arguments are ............. 79
  2.7.13 Explicit displaying of implicit arguments for pretty-printing ... 79
  2.7.14 Interaction with subtyping ................................ 79
  2.7.15 Canonical structures .................................... 80
  2.7.16 Implicit types of variables ............................. 81
2.8 Coercions ....................................................... 82
2.9 Printing constructions in full .................................. 82
2.10 Printing universes ............................................. 82

3 The Coq library .................................................. 83
  3.1 The basic library ............................................. 83
    3.1.1 Notations .............................................. 83
    3.1.2 Logic .................................................. 83
    3.1.3 Datatypes .............................................. 86
    3.1.4 Specification .......................................... 87
    3.1.5 Basic Arithmetics ..................................... 89
    3.1.6 Well-founded recursion ................................ 91
    3.1.7 Accessing the Type level ............................... 92
    3.1.8 Tactics ............................................... 92
Table of contents

3.2 The standard library .............................................. 92
  3.2.1 Survey ....................................................... 92
  3.2.2 Notations for integer arithmetics .......................... 93
  3.2.3 Peano’s arithmetic (nat) .................................... 93
  3.2.4 Real numbers library ........................................ 94
  3.2.5 List library ................................................. 96
3.3 Users’ contributions .............................................. 96

4 Calculus of Inductive Constructions .............................. 97
  4.1 The terms ....................................................... 97
     4.1.1 Sorts ....................................................... 98
     4.1.2 Constants ................................................... 98
     4.1.3 Terms ....................................................... 99
  4.2 Typed terms ..................................................... 100
  4.3 Conversion rules ................................................ 102
  4.4 Derived rules for environments ................................ 103
  4.5 Inductive Definitions ......................................... 104
     4.5.1 Representing an inductive definition ..................... 104
     4.5.2 Types of inductive objects ............................... 107
     4.5.3 Well-formed inductive definitions ....................... 107
     4.5.4 Destructors ............................................... 111
     4.5.5 Fixpoint definitions ..................................... 115
  4.6 Coinductive types .............................................. 119
  4.7 CIC: the Calculus of Inductive Construction with impredicative Set ........................................ 119

5 The Module System .................................................. 121
  5.1 Modules and module types ...................................... 121
  5.2 Typing Modules ............................................... 122

II The proof engine .................................................. 127

6 Vernacular commands .............................................. 129
  6.1 Displaying ..................................................... 129
     6.1.1 Print qualid ............................................... 129
     6.1.2 Print All .................................................... 129
  6.2 Requests to the environment ................................... 130
     6.2.1 Check term ................................................. 130
     6.2.2 Eval convtactic in term ................................... 130
     6.2.3 Compute term ............................................... 130
     6.2.4 Extraction term ............................................ 130
     6.2.5 Print Assumptions qualid .................................. 130
     6.2.6 Search qualid .............................................. 130
     6.2.7 SearchAbout qualid ....................................... 131
     6.2.8 SearchPattern term_pattern ............................... 132
     6.2.9 SearchRewrite term ........................................ 133
     6.2.10 Locate qualid ............................................. 133
     6.2.11 The WHELP searching tool ............................... 134
### Table of contents

6.3 Loading files ................................. 135
   6.3.1 Load ident .............................. 135
6.4 Compiled files ............................... 135
   6.4.1 Require qualid ......................... 136
   6.4.2 Print Libraries ......................... 137
   6.4.3 Declare ML Module string\_1 \ldots string\_n \ldots \ldots \ldots 137
   6.4.4 Print ML Modules ....................... 137
6.5 Loadpath ................................... 137
   6.5.1 Pwd. .................................. 137
   6.5.2 Cd string ............................. 138
   6.5.3 Add LoadPath string as dirpath ....... 138
   6.5.4 Add Rec LoadPath string as dirpath ... 138
   6.5.5 Remove LoadPath string ............... 138
   6.5.6 Print LoadPath ......................... 139
   6.5.7 Add ML Path string .................... 139
   6.5.8 Add Rec ML Path string ............... 139
   6.5.9 Print ML Path string ................... 139
   6.5.10 Locate File string .................... 139
   6.5.11 Locate Library dirpath ................ 139
6.6 States and Reset ............................ 139
   6.6.1 Reset ident ............................ 139
   6.6.2 Back ................................. 140
   6.6.3 Backtrack num\_1 num\_2 num\_3 ........ 140
   6.6.4 Restore State string ................... 141
   6.6.5 Write State string ..................... 141
6.7 Quitting and debugging ..................... 141
   6.7.1 Quit .................................. 141
   6.7.2 Drop .................................. 141
   6.7.3 Time command ......................... 142
6.8 Controlling display ......................... 142
   6.8.1 Set Silent ............................. 142
   6.8.2 Unset Silent ......................... 142
   6.8.3 Set Printing Width integer ........... 142
   6.8.4 Unset Printing Width ................. 142
   6.8.5 Test Printing Width .................... 142
   6.8.6 Set Printing Depth integer ........... 142
   6.8.7 Unset Printing Depth ................. 142
   6.8.8 Test Printing Depth .................... 142
6.9 Controlling the reduction strategies and the conversion algorithm 143
   6.9.1 Opaque qualid\_1 \ldots qualid\_n ........ 143
   6.9.2 Transparent qualid\_1 \ldots qualid\_n .... 143
   6.9.3 Strategy level [ qualid\_1 \ldots qualid\_n ] .... 144
   6.9.4 Set Virtual Machine .................. 144
   6.9.5 Unset Virtual Machine ............... 144
   6.9.6 Test Virtual Machine .................. 144
7 Proof handling

7.1 Switching on/off the proof editing mode

7.1.1 Goal form

7.1.2 Qed.

7.1.3 Admitted.

7.1.4 Theorem ident : form.

7.1.5 Proof term.

7.1.6 Abort.

7.1.7 Suspend.

7.1.8 Resume.

7.1.9 Existential num := term.

7.2 Navigation in the proof tree

7.2.1 Undo.

7.2.2 Set Undo num.

7.2.3 Unset Undo.

7.2.4 Restart.

7.2.5 Focus.

7.2.6 Unfocus.

7.3 Requesting information

7.3.1 Show.

7.3.2 Guarded.

7.3.3 Set Hyps Limit num.

7.3.4 Unset Hyps Limit.

8 Tactics

8.1 Invocation of tactics

8.2 Explicit proof as a term

8.2.1 exact term

8.2.2 refine term

8.3 Basics

8.3.1 assumption

8.3.2 clear ident

8.3.3 move ident_1 after ident_2

8.3.4 rename ident_1 into ident_2

8.3.5 intro

8.3.6 apply term

8.3.7 set ( ident := term )

8.3.8 assert ( ident : form )

8.3.9 apply term in ident

8.3.10 generalize term

8.3.11 change term

8.3.12 fix ident num

8.3.13 cofix ident

8.3.14 evar (ident : term)

8.3.15 instantiate (num := term)

8.3.16 admit

8.3.17 Bindings list
# Table of contents

8.3.18  Occurrences sets and occurrences clauses          .............................................. 166
8.4  Negation and contradiction                                ...................................................... 167
  8.4.1 absurd term                                        ......................................................... 167
  8.4.2 contradiction                                      ......................................................... 167
  8.4.3 contradict ident                                  ......................................................... 167
8.5  Conversion tactics                                        ......................................................... 167
  8.5.1 cbv flag\_1 \ldots flag\_n, lazy flag\_1 \ldots flag\_n and compute ...................... 168
  8.5.2 red                                               ............................................................ 169
  8.5.3 hnf                                               ............................................................ 169
  8.5.4 simpl                                            ............................................................ 169
  8.5.5 unfold qualid                                     .......................................................... 170
  8.5.6 fold term                                         .......................................................... 171
  8.5.7 pattern term                                      .......................................................... 171
  8.5.8 Conversion tactics applied to hypotheses           ................................................. 172
8.6  Introductions                                            ......................................................... 172
  8.6.1 constructor num                                    ......................................................... 172
8.7  Induction and Case Analysis                              ....................................................... 173
  8.7.1 induction term                                     ......................................................... 173
  8.7.2 destruct term                                      ......................................................... 176
  8.7.3 intros intro\_pattern \ldots intro\_pattern          .............................................. 178
  8.7.4 double induction ident\_1 ident\_2                 .................................................... 181
  8.7.5 dependent induction ident                         ....................................................... 182
  8.7.6 decompose [ qualid\_1 \ldots qualid\_n ] term      ............................................. 183
  8.7.7 functional induction (qualid term\_1 \ldots term\_n) ........................................ 184
8.8  Equality                                                ............................................................. 185
  8.8.1 rewrite term                                       ......................................................... 185
  8.8.2 cutrewrite \rightarrow term\_1 = term\_2             ................................................ 186
  8.8.3 replace term\_1 with term\_2                       ....................................................... 187
  8.8.4 reflexivity                                        ......................................................... 187
  8.8.5 symmetry                                          ......................................................... 187
  8.8.6 transitivity term                                  ......................................................... 188
  8.8.7 subst ident                                       .......................................................... 188
  8.8.8 stepl term                                         ......................................................... 188
  8.8.9 f\_equal                                          ............................................................ 188
8.9  Equality and inductive sets                              ....................................................... 189
  8.9.1 decide equality                                    ......................................................... 189
  8.9.2 compare term\_1 term\_2                            ......................................................... 189
  8.9.3 discriminate term                                 ......................................................... 189
  8.9.4 injection term                                    ......................................................... 190
  8.9.5 simplify\_eq term                                  ......................................................... 192
  8.9.6 dependent rewrite \rightarrow ident                .................................................... 192
8.10 Inversion                                               ............................................................. 193
  8.10.1 inversion ident                                   ......................................................... 193
  8.10.2 Derive Inversion ident with \forall (\vec{x:T}).I \vec{t} Sort sort ..................... 195
  8.10.3 functional inversion ident                        ......................................................... 196
  8.10.4 quote ident                                       ......................................................... 196
8.11 Classical tactics                                      ......................................................... 196
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.11.1</td>
<td>classical_left, classical_right</td>
<td>197</td>
</tr>
<tr>
<td>8.12</td>
<td>Automatizing</td>
<td>197</td>
</tr>
<tr>
<td>8.12.1</td>
<td>auto</td>
<td>197</td>
</tr>
<tr>
<td>8.12.2</td>
<td>eauto</td>
<td>198</td>
</tr>
<tr>
<td>8.12.3</td>
<td>tauto</td>
<td>198</td>
</tr>
<tr>
<td>8.12.4</td>
<td>intuition tactic</td>
<td>199</td>
</tr>
<tr>
<td>8.12.5</td>
<td>rtauto</td>
<td>199</td>
</tr>
<tr>
<td>8.12.6</td>
<td>firstorder</td>
<td>200</td>
</tr>
<tr>
<td>8.12.7</td>
<td>congruence</td>
<td>200</td>
</tr>
<tr>
<td>8.12.8</td>
<td>omega</td>
<td>202</td>
</tr>
<tr>
<td>8.12.9</td>
<td>ring and ring_simplify $term_1 \ldots term_n$</td>
<td>202</td>
</tr>
<tr>
<td>8.12.10</td>
<td>field, field_simplify $term_1 \ldots term_n$ and field_simplify_eq</td>
<td>202</td>
</tr>
<tr>
<td>8.12.11</td>
<td>fourier</td>
<td>203</td>
</tr>
<tr>
<td>8.12.12</td>
<td>autorewrite with $ident_1 \ldots ident_n$</td>
<td>203</td>
</tr>
<tr>
<td>8.13</td>
<td>Controlling automation</td>
<td>204</td>
</tr>
<tr>
<td>8.13.1</td>
<td>The hints databases for auto and eauto</td>
<td>204</td>
</tr>
<tr>
<td>8.13.2</td>
<td>Hint databases defined in the Coq standard library</td>
<td>207</td>
</tr>
<tr>
<td>8.13.3</td>
<td>Print Hint</td>
<td>208</td>
</tr>
<tr>
<td>8.13.4</td>
<td>Hint Rewrite $term_1 \ldots term_n : ident$</td>
<td>208</td>
</tr>
<tr>
<td>8.13.5</td>
<td>Hints and sections</td>
<td>209</td>
</tr>
<tr>
<td>8.13.6</td>
<td>Setting implicit automation tactics</td>
<td>209</td>
</tr>
<tr>
<td>8.14</td>
<td>Generation of induction principles with Scheme</td>
<td>210</td>
</tr>
<tr>
<td>8.14.1</td>
<td>Automatic declaration of schemes</td>
<td>210</td>
</tr>
<tr>
<td>8.14.2</td>
<td>Combined Scheme</td>
<td>211</td>
</tr>
<tr>
<td>8.15</td>
<td>Generation of induction principles with Functional Scheme</td>
<td>211</td>
</tr>
<tr>
<td>8.16</td>
<td>Simple tactic macros</td>
<td>211</td>
</tr>
<tr>
<td>9</td>
<td>The tactic language</td>
<td>213</td>
</tr>
<tr>
<td>9.1</td>
<td>Syntax</td>
<td>213</td>
</tr>
<tr>
<td>9.2</td>
<td>Semantics</td>
<td>214</td>
</tr>
<tr>
<td>9.3</td>
<td>Tactic toplevel definitions</td>
<td>223</td>
</tr>
<tr>
<td>9.3.1</td>
<td>Defining $Ltac$ functions</td>
<td>223</td>
</tr>
<tr>
<td>9.3.2</td>
<td>Printing $Ltac$ tactics</td>
<td>224</td>
</tr>
<tr>
<td>9.4</td>
<td>Debugging $Ltac$ tactics</td>
<td>224</td>
</tr>
<tr>
<td>10</td>
<td>Detailed examples of tactics</td>
<td>225</td>
</tr>
<tr>
<td>10.1</td>
<td>refine</td>
<td>225</td>
</tr>
<tr>
<td>10.2</td>
<td>eapply</td>
<td>226</td>
</tr>
<tr>
<td>10.3</td>
<td>Scheme</td>
<td>227</td>
</tr>
<tr>
<td>10.3.1</td>
<td>Combined Scheme</td>
<td>228</td>
</tr>
<tr>
<td>10.4</td>
<td>Functional Scheme and functional induction</td>
<td>229</td>
</tr>
<tr>
<td>10.5</td>
<td>inversion</td>
<td>231</td>
</tr>
<tr>
<td>10.6</td>
<td>dependent induction</td>
<td>234</td>
</tr>
<tr>
<td>10.6.1</td>
<td>A larger example</td>
<td>237</td>
</tr>
<tr>
<td>10.7</td>
<td>autorewrite</td>
<td>240</td>
</tr>
<tr>
<td>10.8</td>
<td>quote</td>
<td>241</td>
</tr>
<tr>
<td>10.8.1</td>
<td>Introducing variables map</td>
<td>242</td>
</tr>
</tbody>
</table>
## Table of contents

10.8.2 Combining variables and constants ......................................................... 243
10.9 Using the tactical language ............................................................................ 245
10.9.1 About the cardinality of the set of natural numbers ................................. 245
10.9.2 Permutation on closed lists ........................................................................... 245
10.9.3 Deciding intuitionistic propositional logic ................................................... 247
10.9.4 Deciding type isomorphisms ....................................................................... 247

11 The C-zar mathematical proof language ............................................................ 253
11.1 Introduction ....................................................................................................... 253
11.1.1 Foreword ....................................................................................................... 253
11.1.2 What is a declarative proof ? ........................................................................ 253
11.1.3 Well-formedness and Completeness .............................................................. 253
11.1.4 Note for tactics users .................................................................................... 254
11.1.5 Compatibility ............................................................................................... 254
11.2 Syntax ............................................................................................................... 254
11.2.1 Temporary names ......................................................................................... 254
11.3 Language description ...................................................................................... 255
11.3.1 Starting and Ending a mathematical proof .................................................. 255
11.3.2 Switching modes ......................................................................................... 257
11.3.3 Computation steps ....................................................................................... 258
11.3.4 Deduction steps ............................................................................................ 259
11.3.5 Iterated equalities ....................................................................................... 260
11.3.6 Subproofs ..................................................................................................... 261
11.3.7 Conclusion steps .......................................................................................... 262
11.3.8 Declaring an Abbreviation ........................................................................ 267
11.3.9 Introduction steps ....................................................................................... 268
11.3.10 Tuple elimination steps ............................................................................. 270
11.3.11 Disjunctive reasoning ............................................................................... 271
11.3.12 Proofs per cases ....................................................................................... 274
11.3.13 Proofs by induction ................................................................................... 275
11.3.14 Justifications .............................................................................................. 276
11.4 More details and Formal Semantics ............................................................... 277

III  User extensions ................................................................................................. 279

12 Syntax extensions and interpretation scopes ..................................................... 281
12.1 Notations ......................................................................................................... 281
12.1.1 Basic notations ............................................................................................ 281
12.1.2 Precedences and associativity ..................................................................... 282
12.1.3 Complex notations ...................................................................................... 282
12.1.4 Simple factorization rules .......................................................................... 283
12.1.5 Displaying symbolic notations .................................................................... 284
12.1.6 The \texttt{Infix} command ........................................................................... 285
12.1.7 Reserving notations .................................................................................... 285
12.1.8 Simultaneous definition of terms and notations ....................................... 286
12.1.9 Displaying informations about notations .................................................. 286
# Table of contents

12.1.10 Locating notations ...................................... 286  
12.1.11 Notations with recursive patterns .......................... 287  
12.1.12 Notations and binders .................................. 288  
12.1.13 Summary .................................................. 289  
12.2 Interpretation scopes ........................................ 289  
12.2.1 Global interpretation rules for notations .................... 289  
12.2.2 Local interpretation rules for notations .................... 290  
12.2.3 The \texttt{type\_scope} interpretation scope ................. 292  
12.2.4 Interpretation scopes used in the standard library of COQ ... 292  
12.2.5 Displaying informations about scopes ....................... 293  
12.3 Abbreviations .............................................. 294  
12.4 Tactic Notations ............................................ 295  

IV Practical tools ............................................. 297  
13 The COQ commands ........................................... 299  
13.1 Interactive use (coqtop) .................................... 299  
13.2 Batch compilation (coqc) .................................... 299  
13.3 Resource file ............................................... 300  
13.4 Environment variables ...................................... 300  
13.5 Options ..................................................... 300  
13.6 Compiled libraries checker (coqchk) .......................... 303  

14 Utilities ..................................................... 305  
14.1 Building a toplevel extended with user tactics .................. 305  
14.2 Modules dependencies ....................................... 306  
14.3 Creating a Makefile for COQ modules ......................... 306  
14.4 Documenting COQ files with coqdoc .......................... 307  
14.4.1 Principles ............................................... 307  
14.4.2 Usage .................................................... 309  
14.4.3 The coqdoc \LaTeX{} style file ............................ 312  
14.5 Exporting COQ theories to XML ................................ 313  
14.5.1 Practical use of the XML exportation tool .................. 313  
14.5.2 Reflection of the logical structure into the file system ... 313  
14.5.3 What is exported? ....................................... 314  
14.5.4 Inner types .............................................. 314  
14.5.5 Interactive exportation commands ......................... 314  
14.5.6 Applications: rendering, searching and publishing ......... 315  
14.5.7 Technical informations .................................. 315  
14.6 Embedded COQ phrases inside \LaTeX{} documents ............... 317  
14.7 COQ and GNU EMACS ........................................ 318  
14.7.1 The COQ Emacs mode ..................................... 318  
14.7.2 Proof General .......................................... 318  
14.8 Module specification ....................................... 318  
14.9 Man pages .................................................. 318
# Table of contents

15 **COQ Integrated Development Environment**  
15.1 Managing files and buffers, basic edition  
15.2 Interactive navigation into COQ scripts  
15.3 Try tactics automatically  
15.4 Vernacular commands, templates  
15.5 Queries  
15.6 Compilation  
15.7 Customizations  
15.8 Using unicode symbols  
15.8.1 Displaying unicode symbols  
15.8.2 Defining an input method for non ASCII symbols  
15.8.3 Character encoding for saved files  
15.9 Building a custom COQIDE with user ML code  

V **Addendum to the Reference Manual**  

16 **Extended pattern-matching**  
16.1 Patterns  
16.2 About patterns of parametric types  
16.3 Matching objects of dependent types  
16.3.1 Understanding dependencies in patterns  
16.3.2 When the elimination predicate must be provided  
16.4 Using pattern matching to write proofs  
16.5 Pattern-matching on inductive objects involving local definitions  
16.6 Pattern-matching and coercions  
16.7 When does the expansion strategy fail?  

17 **Implicit Coercions**  
17.1 General Presentation  
17.2 Classes  
17.3 Coercions  
17.4 Identity Coercions  
17.5 Inheritance Graph  
17.6 Declaration of Coercions  
17.6.1 Coercion *qualid*: class$_1$ $\rightarrow$ class$_2$.  
17.6.2 Identity Coercion *ident*: class$_1$ $\rightarrow$ class$_2$.  
17.7 Displaying Available Coercions  
17.7.1 Print Classes.  
17.7.2 Print Coercions.  
17.7.3 Print Graph.  
17.7.4 Print Coercion Paths class$_1$, class$_2$.  
17.8 Activating the Printing of Coercions  
17.8.1 Set Printing Coercions.  
17.8.2 Set Printing Coercion *qualid*.  
17.9 Classes as Records  
17.10 Coercions and Sections  

Coq Reference Manual, V8.2p12, June 29, 2010
# Coq Reference Manual, V8.2pl2, June 29, 2010

## Table of contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>17.11</td>
<td>Examples</td>
</tr>
<tr>
<td>18</td>
<td>Type Classes</td>
</tr>
<tr>
<td>18.1</td>
<td>Class and Instance declarations</td>
</tr>
<tr>
<td>18.2</td>
<td>Binding classes</td>
</tr>
<tr>
<td>18.2.1</td>
<td>Implicit quantification</td>
</tr>
<tr>
<td>18.3</td>
<td>Parameterized Instances</td>
</tr>
<tr>
<td>18.4</td>
<td>Building hierarchies</td>
</tr>
<tr>
<td>18.4.1</td>
<td>Superclasses</td>
</tr>
<tr>
<td>18.4.2</td>
<td>Substructures</td>
</tr>
<tr>
<td>18.5</td>
<td>Summary of the commands</td>
</tr>
<tr>
<td>18.5.1</td>
<td>Class (\text{ident} \ \text{binder}_1 \ldots \text{binder}_n : \text{sort} := { \text{field}_1 ; \ldots ; \text{field}_k } )</td>
</tr>
<tr>
<td>18.5.2</td>
<td>Instance (\text{ident} \ \text{binder}_1 \ldots \text{binder}_n : \text{Class } t_1 \ldots t_n [</td>
</tr>
<tr>
<td>18.5.3</td>
<td>Existing Instance (\text{ident} )</td>
</tr>
<tr>
<td>18.5.4</td>
<td>Typeclasses Transparent, Opaque (\text{ident}_1 \ldots \text{ident}_n )</td>
</tr>
<tr>
<td>18.5.5</td>
<td>Typeclasses eauto := [debug] [dfs</td>
</tr>
<tr>
<td>19</td>
<td>Omega: a solver of quantifier-free problems in Presburger Arithmetic</td>
</tr>
<tr>
<td>19.1</td>
<td>Description of omega</td>
</tr>
<tr>
<td>19.1.1</td>
<td>Arithmetical goals recognized by omega</td>
</tr>
<tr>
<td>19.1.2</td>
<td>Messages from omega</td>
</tr>
<tr>
<td>19.2</td>
<td>Using omega</td>
</tr>
<tr>
<td>19.3</td>
<td>Technical data</td>
</tr>
<tr>
<td>19.3.1</td>
<td>Overview of the tactic</td>
</tr>
<tr>
<td>19.3.2</td>
<td>Overview of the OMEGA decision procedure</td>
</tr>
<tr>
<td>19.4</td>
<td>Bugs</td>
</tr>
<tr>
<td>20</td>
<td>Micromega: tactics for solving arithmetics goals over ordered rings</td>
</tr>
<tr>
<td>20.1</td>
<td>The psatz tactic in a hurry</td>
</tr>
<tr>
<td>20.2</td>
<td>Positivstellensatz refutations</td>
</tr>
<tr>
<td>20.3</td>
<td>lia: the linear integer arithmetic tactic</td>
</tr>
<tr>
<td>21</td>
<td>Extraction of programs in Objective Caml and Haskell</td>
</tr>
<tr>
<td>21.1</td>
<td>Generating ML code</td>
</tr>
<tr>
<td>21.2</td>
<td>Extraction options</td>
</tr>
<tr>
<td>21.2.1</td>
<td>Setting the target language</td>
</tr>
<tr>
<td>21.2.2</td>
<td>Inlining and optimizations</td>
</tr>
<tr>
<td>21.2.3</td>
<td>Realizing axioms</td>
</tr>
<tr>
<td>21.2.4</td>
<td>Avoiding conflicts with existing filenames</td>
</tr>
<tr>
<td>21.3</td>
<td>Differences between COQ and ML type systems</td>
</tr>
<tr>
<td>21.4</td>
<td>Some examples</td>
</tr>
<tr>
<td>21.4.1</td>
<td>A detailed example: Euclidean division</td>
</tr>
<tr>
<td>21.4.2</td>
<td>Another detailed example: Heapsort</td>
</tr>
<tr>
<td>21.4.3</td>
<td>The Standard Library</td>
</tr>
<tr>
<td>21.4.4</td>
<td>Extraction’s horror museum</td>
</tr>
<tr>
<td>21.4.5</td>
<td>Users’ Contributions</td>
</tr>
</tbody>
</table>
# Table of contents

22 Program

22.1 Elaborating programs

- 22.1.1 Program Definition \( \textit{ident} \ := \ \textit{term} \)
- 22.1.2 Program Fixpoint \( \textit{ident} \ \textit{params} \ \{\textit{order}\} : \ \textit{type} \ := \ \textit{term} \)
- 22.1.3 Program Lemma \( \textit{ident} : \ \textit{type} \)

22.2 Solving obligations

22.3 Frequently Asked Questions

23 The ring and field tactic families

- 23.1 What does this tactic do?

- 23.2 The variables map

- 23.3 Is it automatic?

- 23.4 Concrete usage in Coq

- 23.5 Adding a ring structure

- 23.6 How does it work?

- 23.7 Dealing with fields

- 23.8 Adding a new field structure

- 23.9 Legacy implementation

- 23.9.1 legacy ring \( \textit{term}_1 \ldots \textit{term}_n \)
- 23.9.2 Add a ring structure
- 23.9.3 legacy field
- 23.9.4 Add Legacy Field
- 23.10 History of ring

- 23.11 Discussion

24 User defined equalities and relations

- 24.1 Relations and morphisms

- 24.2 Adding new relations and morphisms

- 24.3 Rewriting and non reflexive relations

- 24.4 Rewriting and non symmetric relations

- 24.5 Rewriting in ambiguous setoid contexts

- 24.6 First class setoids and morphisms

- 24.7 Tactics enabled on user provided relations

- 24.8 Printing relations and morphisms

- 24.9 Deprecated syntax and backward incompatibilities

- 24.10 Rewriting under binders

- 24.11 Sub-relations

- 24.12 Constant unfolding

25 Calling external provers

- 25.1 The gappa tactic

Bibliography

Global Index

Tactics Index
## Table of contents

<table>
<thead>
<tr>
<th>Vernacular Commands Index</th>
<th>431</th>
</tr>
</thead>
<tbody>
<tr>
<td>Index of Error Messages</td>
<td>435</td>
</tr>
</tbody>
</table>
Part I

The language
Chapter 1

The GALLINA specification language

This chapter describes GALLINA, the specification language of COQ. It allows to develop mathematical theories and to prove specifications of programs. The theories are built from axioms, hypotheses, parameters, lemmas, theorems and definitions of constants, functions, predicates and sets. The syntax of logical objects involved in theories is described in Section 1.2. The language of commands, called The Vernacular is described in section 1.3.

In COQ, logical objects are typed to ensure their logical correctness. The rules implemented by the typing algorithm are described in Chapter 4.

About the grammars in the manual

Grammars are presented in Backus-Naur form (BNF). Terminal symbols are set in typewriter font. In addition, there are special notations for regular expressions.

An expression enclosed in square brackets [...] means at most one occurrence of this expression (this corresponds to an optional component).

The notation “entry sep … sep entry” stands for a non empty sequence of expressions parsed by entry and separated by the literal “sep”¹.

Similarly, the notation “entry … entry” stands for a non empty sequence of expressions parsed by the “entry” entry, without any separator between.

At the end, the notation “[entry sep … sep entry]” stands for a possibly empty sequence of expressions parsed by the “entry” entry, separated by the literal “sep”.

1.1 Lexical conventions

Blanks Space, newline and horizontal tabulation are considered as blanks. Blanks are ignored but they separate tokens.

Comments Comments in COQ are enclosed between (* and *), and can be nested. They can contain any character. However, string literals must be correctly closed. Comments are treated as blanks.

Identifiers and access identifiers Identifiers, written ident, are sequences of letters, digits, _ and ’, that do not start with a digit or ’. That is, they are recognized by the following lexical class:

¹This is similar to the expression “entry { sep entry }” in standard BNF, or “entry ( sep entry )∗” in the syntax of regular expressions.
The Gallina specification language

\[\begin{align*}
\text{first_letter} & ::= \text{a..z|A..Z|_|unicode-letter} \\
\text{subsequent_letter} & ::= \text{a..z|A..Z|0..9|’|unicode-letter|unicode-id-part} \\
\text{ident} & ::= \text{first_letter[subsequent_letter…subsequent_letter]}
\end{align*}\]

All characters are meaningful. In particular, identifiers are case-sensitive. The entry \text{unicode-letter} non-exhaustively includes Latin, Greek, Gothic, Cyrillic, Arabic, Hebrew, Georgian, Hangul, Hiragana and Katakana characters, CJK ideographs, mathematical letter-like symbols, hyphens, non-breaking space, ... The entry \text{unicode-id-part} non-exhaustively includes symbols for prime letters and subscripts.

Access identifiers, written \text{access_ident}, are identifiers prefixed by . (dot) without blank. They are used in the syntax of qualified identifiers.

**Natural numbers and integers**  Numerals are sequences of digits. Integers are numerals optionally preceded by a minus sign.

\[\begin{align*}
\text{digit} & ::= \text{0..9} \\
\text{num} & ::= \text{digit…digit} \\
\text{integer} & ::= [\text{-}\text{num}]
\end{align*}\]

**Strings**  Strings are delimited by " (double quote), and enclose a sequence of any characters different from " or the sequence "\"" to denote the double quote character. In grammars, the entry for quoted strings is \text{string}.

**Keywords**  The following identifiers are reserved keywords, and cannot be employed otherwise:

\[
_\quad\text{as}\quad\text{at}\quad\text{cofix}\quad\text{else}\quad\text{end}
\] \[
\text{exists}\quad\text{exists2}\quad\text{fix}\quad\text{for}\quad\text{forall}\quad\text{fun}
\] \[
\text{if}\quad\text{IF}\quad\text{in}\quad\text{let}\quad\text{match}\quad\text{mod}
\] \[
\text{Prop}\quad\text{return}\quad\text{Set}\quad\text{then}\quad\text{Type}\quad\text{using}
\]
\[
\text{where}\quad\text{with}
\]

**Special tokens**  The following sequences of characters are special tokens:

\[
!\quad\%\quad\&\quad\&\&\quad(\quad(\quad)
\]
\[
*\quad+\quad++\quad,\quad-\quad->\quad.
\]
\[
.(\quad..\quad/\quad\:\quad::\quad::<
\]
\[
::=\quad>:\quad;\quad<\quad<-\quad<->\quad<:
\]
\[
<=\quad<>\quad=\quad>=\quad=>\quad=_\text{D}\quad>\quad>>
\]
\[
>=\quad?\quad?=\quad@\quad[\quad\:\quad\]
\]
\[
^\quad\{\quad\mid\quad\\mid-\quad\mid\quad\}\quad~
\]

Lexical ambiguities are resolved according to the "longest match" rule: when a sequence of non alphanumerical characters can be decomposed into several different ways, then the first token is the longest possible one (among all tokens defined at this moment), and so on.
1.2 Terms

1.2.1 Syntax of terms

Figures 1.1 and 1.2 describe the basic set of terms which form the Calculus of Inductive Constructions (also called pCIC). The formal presentation of pCIC is given in Chapter 4. Extensions of this syntax are given in chapter 2. How to customize the syntax is described in Chapter 12.

1.2.2 Types

COQ terms are typed. COQ types are recognized by the same syntactic class as term. We denote by type the semantic subclass of types inside the syntactic class term.

1.2.3 Qualified identifiers and simple identifiers

Qualified identifiers (qualid) denote global constants (definitions, lemmas, theorems, remarks or facts), global variables (parameters or axioms), inductive types or constructors of inductive types. Simple identifiers (or shortly ident) are a syntactic subset of qualified identifiers. Identifiers may also denote local variables, what qualified identifiers do not.

1.2.4 Numerals

Numerals have no definite semantics in the calculus. They are mere notations that can be bound to objects through the notation mechanism (see Chapter 12 for details). Initially, numerals are bound to Peano’s representation of natural numbers (see 3.1.3).

Note: negative integers are not at the same level as num, for this would make precedence unnatural.

1.2.5 Sorts

There are three sorts Set, Prop and Type.

- Prop is the universe of logical propositions. The logical propositions themselves are typing the proofs. We denote propositions by form. This constitutes a semantic subclass of the syntactic class term.

- Set is the universe of program types or specifications. The specifications themselves are typing the programs. We denote specifications by specif. This constitutes a semantic subclass of the syntactic class term.

- Type is the type of Set and Prop

More on sorts can be found in Section 4.1.1.

COQ terms are typed. COQ types are recognized by the same syntactic class as term. We denote by type the semantic subclass of types inside the syntactic class term.
The **GALLINA** specification language

```
term ::= forall binderlist , term (1.2.8)
  |  fun binderlist => term (1.2.7)
  |  fix fix_bodies (1.2.14)
  |  cofix cofix_bodies (1.2.14)
  |  let ident_with_params := term in term (1.2.12)
  |  let fix fix_body in term (1.2.14)
  |  let cofix cofix_body in term (1.2.14)
  |  let ( [name , . . . , name] ) [dep_ret_type] := term in term (1.2.13, 2.2.1)
  |  if term [dep_ret_type] then term else term (1.2.13, 2.2.1)
  |  term : term (1.2.10)
  |  term -> term (1.2.8)
  |  term arg . . . arg (1.2.9)
  |  @ qualid [term . . . term] (2.7.11)
  |  term % ident (12.2.2)
  |  match match_item , . . . , match_item [return_type] with
    [ [ ] equation | . . . | equation ] end (1.2.13)
  |  qualid (1.2.3)
  |  sort (1.2.5)
  |  num (1.2.4)
  |  _ (1.2.11)

arg ::= term (2.7.11)
  | ( ident := term )

binderlist ::= name . . . name [: term] 1.2.6
  |  binder binderlet . . . binderlet

binder ::= name 1.2.6
  | ( name . . . name : term )

binderlet ::= binder 1.2.6
  | ( name [: term] := term )

name ::= ident
  | _

qualid ::= ident
  | qualid access_ident

sort ::= Prop | Set | Type
```

Figure 1.1: Syntax of terms

**1.2.6 Binders**

Various constructions such as `fun`, `forall`, `fix` and `cofix bind` variables. A binding is represented by an identifier. If the binding variable is not used in the expression, the identifier can be replaced by the
ident_with_params ::= ident [binderlet ... binderlet] [: term]

fix_bodies ::= fix_body
| fix_body with fix_body with ... with fix_body for ident

cofix_bodies ::= cofix_body
| cofix_body with cofix_body with ... with cofix_body for ident

fix_body ::= ident binderlet ... binderlet [annotation] [: term] := term

cofix_body ::= ident_with_params := term

annotation ::= { struct ident }

match_item ::= term [as name] [in term]

dep_ret_type ::= [as name] return_type

return_type ::= return term

equation ::= mult_pattern | ... | mult_pattern => term

mult_pattern ::= pattern, ..., pattern

pattern ::= qualid pattern ... pattern
| pattern as ident
| pattern % ident
| qualid
| _
| num
| ( or_pattern , ... , or_pattern )

or_pattern ::= pattern | ... | pattern

symbol _. When the type of a bound variable cannot be synthesized by the system, it can be specified with the notation (ident : type). There is also a notation for a sequence of binding variables sharing the same type: ( ident₁...identₙ : type ).

Some constructions allow the binding of a variable to a value. This is called a “let-binder”. The entry binderlet of the grammar accepts either a binder as defined above or a let-binder. The notation in the latter case is (ident := term). In a let-binder, only one variable can be introduced at the same time. It is also possible to give the type of the variable as follows: (ident : term := term).

Lists of binderlet are allowed. In the case of fun and forall, the first binder of the list cannot be a let-binder, but parentheses can be omitted in the case of a single sequence of bindings sharing the same type (e.g.: fun (x y z : A) => t can be shortened in fun x y z : A => t).
1.2.7 Abstractions

The expression “fun ident : type => term” defines the abstraction of the variable ident, of type type, over the term term. It denotes a function of the variable ident that evaluates to the expression term (e.g. fun x:A => x denotes the identity function on type A). The keyword fun can be followed by several binders as given in Section 1.2.6. Functions over several variables are equivalent to an iteration of one-variable functions. For instance the expression “fun ident_1 ... ident_n : type => term” denotes the same function as “fun ident_1 : type => ... fun ident_n : type => term”. If a let-binder occurs in the list of binders, it is expanded to a local definition (see Section 1.2.12).

1.2.8 Products

The expression “forall ident : type, term” denotes the product of the variable ident of type type, over the term term. As for abstractions, forall is followed by a binder list, and products over several variables are equivalent to an iteration of one-variable products. Note that term is intended to be a type.

If the variable ident occurs in term, the product is called dependent product. The intention behind a dependent product forall x:A, B is twofold. It denotes either the universal quantification of the variable x of type A in the proposition B or the functional dependent product from A to B (a construction usually written Πx:A.B in set theory).

Non dependent product types have a special notation: “A => B” stands for “forall _:A, B”. The non dependent product is used both to denote the propositional implication and function types.

1.2.9 Applications

The expression term_0 term_1 denotes the application of term_0 to term_1.

The expression term_0 term_1 ... term_n denotes the application of the term term_0 to the arguments term_1 ... then term_n. It is equivalent to (... (term_0 term_1) ...) term_n: associativity is to the left.

The notation (ident := term) for arguments is used for making explicit the value of implicit arguments (see Section 2.7.11).

1.2.10 Type cast

The expression “term : type” is a type cast expression. It enforces the type of term to be type.

1.2.11 Inferable subterms

Expressions often contain redundant pieces of information. Subterms that can be automatically inferred by Coq can be replaced by the symbol “_” and Coq will guess the missing piece of information.

1.2.12 Local definitions (let-in)

let ident := term_1 in term_2 denotes the local binding of term_1 to the variable ident in term_2. There is a syntactic sugar for local definition of functions: let ident binder_1 ... binder_n := term_1 in term_2 stands for let ident := fun binder_1 ... binder_n => term_2 in term_2.
1.2 Terms

1.2.13 Definition by case analysis

Objects of inductive types can be destructurated by a case-analysis construction called pattern-matching expression. A pattern-matching expression is used to analyze the structure of an inductive objects and to apply specific treatments accordingly.

This paragraph describes the basic form of pattern-matching. See Section 2.2.1 and Chapter 16 for the description of the general form. The basic form of pattern-matching is characterized by a single match_item expression, a mult_pattern restricted to a single pattern and pattern restricted to the form qualid ident ... ident.

The expression match term_0 return_type with pattern_1 => term_1 | ... | pattern_n => term_n end, denotes a pattern-matching over the term term_0 (expected to be of an inductive type I). The terms term_1...term_n are the branches of the pattern-matching expression. Each of pattern_i has a form qualid ident ... ident where qualid must denote a constructor. There should be exactly one branch for every constructor of I.

The return_type expresses the type returned by the whole match expression. There are several cases. In the non dependent case, all branches have the same type, and the return_type is the common type of branches. In this case, return_type can usually be omitted as it can be inferred from the type of the branches^2.

In the dependent case, there are three subcases. In the first subcase, the type in each branch may depend on the exact value being matched in the branch. In this case, the whole pattern-matching itself depends on the term being matched. This dependency of the term being matched in the return type is expressed with an “as ident” clause where ident is dependent in the return type. For instance, in the following example:

Coq < Inductive bool : Type := true : bool | false : bool.
Coq < Inductive eq (A:Type) (x:A) : A -> Prop := refl_equal : eq A x x.
Coq < Inductive or (A:Prop) (B:Prop) : Prop :=
| or_introl : A -> or A B
| or_intror : B -> or A B.
Coq < Definition bool_case (b:bool) : or (eq bool b true) (eq bool b false)
:= match b as x return or (eq bool x true) (eq bool x false) with
| true => or_introl (eq bool true true) (eq bool true false)
| false => or_intror (eq bool false true) (eq bool false false)
end.

the branches have respective types or (eq bool true true) (eq bool true false) and or (eq bool false true) (eq bool false false) while the whole pattern-matching expression has type or (eq bool b true) (eq bool b false), the identifier x being used to represent the dependency. Remark that when the term being matched is a variable, the as clause can be omitted and the term being matched can serve itself as binding name in the return type. For instance, in the following alternative definition is accepted and has the same meaning as the previous one.

Coq < Definition bool_case (b:bool) : or (eq bool b true) (eq bool b false)
:= match b return or (eq bool b true) (eq bool b false) with
| true => or_introl (eq bool true true) (eq bool true false)
| false => or_intror (eq bool false true) (eq bool false false)

^2Except if the inductive type is empty in which case there is no equation to help to infer the return type.
The second subcase is only relevant for annotated inductive types such as the equality predicate (see Section 3.1.2), the order predicate on natural numbers or the type of lists of a given length (see Section 16.3). In this configuration, the type of each branch can depend on the type dependencies specific to the branch and the whole pattern-matching expression has a type determined by the specific dependencies in the type of the term being matched. This dependency of the return type in the annotations of the inductive type is expressed using a "\texttt{in I _ \ldots _ ident_1 \ldots ident_n}" clause, where

- $I$ is the inductive type of the term being matched;
- the names $\texttt{ident}_i$'s correspond to the arguments of the inductive type that carry the annotations: the return type is dependent on them;
- the `_`'s denote the family parameters of the inductive type: the return type is not dependent on them.

For instance, in the following example:

\begin{verbatim}
Coq < Definition sym_equal (A:Type) (x y:A) (H:eq A x y) : eq A y x :=
Coq < match H in eq _ _ z return eq A z x with
Coq < | refl_equal => refl_equal A x
Coq < end.  
\end{verbatim}

the type of the branch has type $\text{eq A x x}$ because the third argument of $\text{eq}$ is $x$ in the type of the pattern $\text{refl_equal}$. On the contrary, the type of the whole pattern-matching expression has type $\text{eq A y x}$ because the third argument of $\text{eq}$ is $y$ in the type of $H$. This dependency of the case analysis in the third argument of $\text{eq}$ is expressed by the identifier $z$ in the return type.

Finally, the third subcase is a combination of the first and second subcase. In particular, it only applies to pattern-matching on terms in a type with annotations. For this third subcase, both the clauses \texttt{as} and \texttt{in} are available.

There are specific notations for case analysis on types with one or two constructors: "\texttt{if ... then ... else ...}" and "\texttt{let ( ... , ... , ...) := ... in ...}" (see Sections 2.2.2 and 2.2.3).

### 1.2.14 Recursive functions

The expression "\texttt{fix ident_1 binder_1 : type_1 := term_1 with ... with ident_n binder_n : type_n := term_n for ident_i}" denotes the $i^{th}$ component of a block of functions defined by mutual well-founded recursion. It is the local counterpart of the \texttt{Fixpoint} command. See Section 1.3.4 for more details. When $n = 1$, the "\texttt{for ident_i}" clause is omitted.

The expression "\texttt{cofix ident_1 binder_1 : type_1 with ... with ident_n binder_n : type_n for ident_i}" denotes the $i^{th}$ component of a block of terms defined by mutual guarded co-recursion. It is the local counterpart of the \texttt{CoFixpoint} command. See Section 1.3.4 for more details. When $n = 1$, the "\texttt{for ident_i}" clause is omitted.

The association of a single fixpoint and a local definition have a special syntax: "\texttt{let fix f ... := ... in ...}" stands for "\texttt{let f := fix f ... := ... in ...}". The same applies for co-fixpoints.
### The Vernacular

1.3 The Vernacular

Figure 1.3 describes The Vernacular which is the language of commands of GALLINA. A sentence of the vernacular language, like in many natural languages, begins with a capital letter and ends with a dot.

The different kinds of command are described hereafter. They all suppose that the terms occurring in the sentences are well-typed.
1.3.1 Declarations

The declaration mechanism allows the user to specify his own basic objects. Declared objects play the role of axioms or parameters in mathematics. A declared object is an \textit{ident} associated to a \textit{term}. A declaration is accepted by Coq if and only if this \textit{term} is a correct type in the current context of the declaration and \textit{ident} was not previously defined in the same module. This \textit{term} is considered to be the type, or specification, of the \textit{ident}.

\textbf{Axiom} \textit{ident} : \textit{term} .

This command links \textit{term} to the name \textit{ident} as its specification in the global context. The fact asserted by \textit{term} is thus assumed as a postulate.

\textbf{Error messages:}

1. \textit{ident} already exists

\textbf{Variants:}

1. \textbf{Parameter} \textit{ident} : \textit{term}.
   Is equivalent to \textbf{Axiom} \textit{ident} : \textit{term}

2. \textbf{Parameter} \textit{ident}_1 \ldots \textit{ident}_n : \textit{term}.
   Adds \textit{n} parameters with specification \textit{term}

3. \textbf{Parameter} (\textit{ident}_{1,1} \ldots \textit{ident}_{1,k_1} : \textit{term}_1) \ldots (\textit{ident}_{n,1} \ldots \textit{ident}_{n,k_n} : \textit{term}_n).
   Adds \textit{n} blocks of parameters with different specifications.

4. \textbf{Conjecture} \textit{ident} : \textit{term}.
   Is equivalent to \textbf{Axiom} \textit{ident} : \textit{term}.

\textbf{Remark:} It is possible to replace \textbf{Parameter} by \textbf{Parameters}.

\textbf{Variable} \textit{ident} : \textit{term}.

This command links \textit{term} to the name \textit{ident} in the context of the current section (see Section 2.4 for a description of the section mechanism). When the current section is closed, name \textit{ident} will be unknown and every object using this variable will be explicitly parametrized (the variable is \textit{discharged}). Using the \textbf{Variable} command out of any section is equivalent to \textbf{Axiom}.

\textbf{Error messages:}

1. \textit{ident} already exists

\textbf{Variants:}

1. \textbf{Variable} \textit{ident}_1 \ldots \textit{ident}_n : \textit{term}.
   Links \textit{term} to names \textit{ident}_1 \ldots \textit{ident}_n.

2. \textbf{Variable} (\textit{ident}_{1,1} \ldots \textit{ident}_{1,k_1} : \textit{term}_1) \ldots (\textit{ident}_{n,1} \ldots \textit{ident}_{n,k_n} : \textit{term}_n).
   Adds \textit{n} blocks of variables with different specifications.
3. Hypothesis \( \text{ident} : \text{term} \).

Hypothesis is a synonymous of Variable.

**Remark:** It is possible to replace Variable by Variables and Hypothesis by Hypotheses. It is advised to use the keywords Axiom and Hypothesis for logical postulates (i.e. when the assertion \( \text{term} \) is of sort Prop), and to use the keywords Parameter and Variable in other cases (corresponding to the declaration of an abstract mathematical entity).

### 1.3.2 Definitions

Definitions differ from declarations in allowing to give a name to a term whereas declarations were just giving a type to a name. That is to say that the name of a defined object can be replaced at any time by its definition. This replacement is called \( \delta \)-conversion (see Section 4.3). A defined object is accepted by the system if and only if the defining term is well-typed in the current context of the definition. Then the type of the name is the type of term. The defined name is called a constant and one says that the constant is added to the environment.

A formal presentation of constants and environments is given in Section 4.2.

Definition \( \text{ident} := \text{term} \).

This command binds the value \( \text{term} \) to the name \( \text{ident} \) in the environment, provided that \( \text{term} \) is well-typed.

**Error messages:**

1. \( \text{ident} \) already exists

**Variants:**

1. **Definition** \( \text{ident} : \text{term}_1 := \text{term}_2 \).

   It checks that the type of \( \text{term}_2 \) is definitionally equal to \( \text{term}_1 \), and registers \( \text{ident} \) as being of type \( \text{term}_1 \), and bound to value \( \text{term}_2 \).

2. **Definition** \( \text{ident binder}_1 \ldots \text{binder}_n : \text{term}_1 := \text{term}_2 \).

   This is equivalent to

   Definition \( \text{ident} : \forall \text{binder}_1 \ldots \text{binder}_n, \text{term}_1 := \text{fun binder}_1 \ldots \text{binder}_n => \text{term}_2 \).

3. **Example** \( \text{ident} := \text{term} \).

   Example \( \text{ident} : \text{term}_1 := \text{term}_2 \).

   Example \( \text{ident binder}_1 \ldots \text{binder}_n : \text{term}_1 := \text{term}_2 \).

   These are synonyms of the Definition forms.

**Error messages:**

1. Error: The term "\( \text{term} \)" has type "\( \text{type} \)" while it is expected to have type "\( \text{type} \)"

**See also:** Sections 6.9.1, 6.9.2, 8.5.5
Let \( \text{ident} := \text{term} \).

This command binds the value \( \text{term} \) to the name \( \text{ident} \) in the environment of the current section. The name \( \text{ident} \) disappears when the current section is eventually closed, and, all persistent objects (such as theorems) defined within the section and depending on \( \text{ident} \) are prefixed by the local definition \( \text{let ident := term in} \).

**Error messages:**

1. \( \text{ident already exists} \)

**Variants:**

1. Let \( \text{ident : term1 := term2} \).

**See also:** Sections 2.4 (section mechanism), 6.9.1, 6.9.2 (opaque/transparent constants), 8.5.5

### 1.3.3 Inductive definitions

We gradually explain simple inductive types, simple annotated inductive types, simple parametric inductive types, mutually inductive types. We explain also co-inductive types.

#### Simple inductive types

The definition of a simple inductive type has the following form:

\[
\text{Inductive ident : sort} := \\
\quad \text{ident}_1 : \text{type}_1 \\
\quad \ldots \\
\quad \text{ident}_n : \text{type}_n
\]

The name \( \text{ident} \) is the name of the inductively defined type and \( \text{sort} \) is the universes where it lives. The names \( \text{ident}_1, \ldots, \text{ident}_n \) are the names of its constructors and \( \text{type}_1, \ldots, \text{type}_n \) their respective types. The types of the constructors have to satisfy a positivity condition (see Section 4.5.3) for \( \text{ident} \). This condition ensures the soundness of the inductive definition. If this is the case, the constants \( \text{ident}, \text{ident}_1, \ldots, \text{ident}_n \) are added to the environment with their respective types. Accordingly to the universe where the inductive type lives (e.g. its type \( \text{sort} \)), COQ provides a number of destructors for \( \text{ident} \). Destructors are named \( \text{ident}_\text{ind}, \text{ident}_\text{rec} \) or \( \text{ident}_\text{rect} \) which respectively correspond to elimination principles on \( \text{Prop}, \text{Set} \) and \( \text{Type} \). The type of the destructors expresses structural induction/recursion principles over objects of \( \text{ident} \). We give below two examples of the use of the Inductive definitions.

The set of natural numbers is defined as:

\[
\text{Coq < Inductive nat : Set :=} \\
\text{Coq < | O : nat} \\
\text{Coq < | S : nat -> nat.} \\
\text{nat is defined} \\
\text{nat_ind is defined} \\
\text{nat_rect is defined} \\
\text{nat_rec is defined}
\]
The type $\text{nat}$ is defined as the least $\text{Set}$ containing $\text{O}$ and closed by the $\text{S}$ constructor. The constants $\text{nat}, \text{O}$ and $\text{S}$ are added to the environment.

Now let us have a look at the elimination principles. They are three of them: $\text{nat\_ind}, \text{nat\_rec}$ and $\text{nat\_rect}$. The type of $\text{nat\_ind}$ is:

Coq < Check nat_ind.

$\text{nat\_ind} : \forall P : \text{nat} \to \text{Prop},$  
$\text{nat\_ind} \text{ P O} \to (\forall n : \text{nat}, \text{ P n} \to \text{ P (S n)}) \to \forall n : \text{nat}, \text{ P n}$

This is the well known structural induction principle over natural numbers, i.e. the second-order form of Peano's induction principle. It allows to prove some universal property of natural numbers ($\forall n: \text{nat}, \text{ P n}$) by induction on $n$.

The types of $\text{nat\_rec}$ and $\text{nat\_rect}$ are similar, except that they pertain to $(P:\text{nat}\to\text{Set})$ and $(P:\text{nat}\to\text{Type})$ respectively. They correspond to primitive induction principles (allowing dependent types) respectively over sorts $\text{Set}$ and $\text{Type}$. The constant $\text{ident\_ind}$ is always provided, whereas $\text{ident\_rec}$ and $\text{ident\_rect}$ can be impossible to derive (for example, when $\text{ident}$ is a proposition).

**Variants:**


   In the case where inductive types have no annotations (next section gives an example of such annotations), a constructor can be defined by only giving the type of its arguments.

**Simple annotated inductive types**

In an annotated inductive types, the universe where the inductive type is defined is no longer a simple sort, but what is called an arity, which is a type whose conclusion is a sort.

As an example of annotated inductive types, let us define the $\text{even}$ predicate:

Coq < Inductive even : nat \to Prop :=
Coq < | even_0 : even O
Coq < | even_SS : forall n:nat, even n \to even (S (S n)).

$\text{even}$ is defined
$\text{even\_ind}$ is defined

The type $\text{nat}\to\text{Prop}$ means that $\text{even}$ is a unary predicate (inductively defined) over natural numbers. The type of its two constructors are the defining clauses of the predicate $\text{even}$. The type of $\text{even\_ind}$ is:

Coq < Check even_ind.

$\text{even\_ind} : \forall P : \text{nat} \to \text{Prop},$

$\text{even\_ind} \text{ P O} \to$

$(\forall n : \text{nat}, \text{ even n} \to \text{ P n} \to \text{ P (S (S n))}) \to$

$\forall n : \text{nat}, \text{ even n} \to \text{ P n}$

From a mathematical point of view it asserts that the natural numbers satisfying the predicate $\text{even}$ are exactly in the smallest set of naturals satisfying the clauses $\text{even\_0}$ or $\text{even\_SS}$. This is why, when we want to prove any predicate $P$ over elements of $\text{even}$, it is enough to prove it for $\text{O}$ and to
prove that if any natural number \( n \) satisfies \( P \) its double successor \( (S \ (S \ n)) \) satisfies also \( P \). This is indeed analogous to the structural induction principle we got for \( \text{nat} \).

**Error messages:**

1. Non strictly positive occurrence of \( \text{id} \) in \( \text{type} \)

2. The conclusion of \( \text{type} \) is not valid; it must be built from \( \text{id} \)

**Parametrized inductive types**

In the previous example, each constructor introduces a different instance of the predicate \( \text{even} \). In some cases, all the constructors introduce the same generic instance of the inductive definition, in which case, instead of an annotation, we use a context of parameters which are binders shared by all the constructors of the definition.

The general scheme is:

\[
\text{Inductive } \text{id} \ \text{binder}_1 \ldots \text{binder}_k : \text{term} := \text{id}_1 : \text{term}_1 | \ldots | \text{id}_n : \text{term}_n.
\]

Parameters differ from inductive type annotations in the fact that the conclusion of each type of constructor \( \text{term}_i \) invoke the inductive type with the same values of parameters as its specification.

A typical example is the definition of polymorphic lists:

Coq < Inductive list (A:Set) : Set :=
Coq < | nil : list A
Coq < | cons : A -> list A -> list A.

Note that in the type of \( \text{nil} \) and \( \text{cons} \), we write \((\text{list} \ A)\) and not just \(\text{list}\).

The constants \( \text{nil} \) and \( \text{cons} \) will have respectively types:

Coq < Check nil.
\[
\text{nil} : \forall A : \text{Set}, \text{list} A
\]

Coq < Check cons.
\[
\text{cons} : \forall A : \text{Set}, A \to \text{list} A \to \text{list} A
\]

Types of destructors are also quantified with \((A:\text{Set})\).

**Variants:**


   This is an alternative definition of lists where we specify the arguments of the constructors rather than their full type.

**Error messages:**

1. The \( \text{num} \)th argument of \( \text{id} \) must be \( \text{id}’ \) in \( \text{type} \)
New from Coq V8.1  The condition on parameters for inductive definitions has been relaxed since Coq V8.1. It is now possible in the type of a constructor, to invoke recursively the inductive definition on an argument which is not the parameter itself.

One can define:

Coq < Inductive list2 (A:Set) : Set :=
Coq <  | nil2 : list2 A
Coq <  | cons2 : A -> list2 (A*A) -> list2 A.
list2 is defined
list2_rect is defined
list2_ind is defined
list2_rec is defined

that can also be written by specifying only the type of the arguments:

Coq < Inductive list2 (A:Set) : Set := nil2 | cons2 (_:A) (_:list2 (A*A)).

But the following definition will give an error:

Coq < Inductive listw (A:Set) : Set :=
Coq <  | nilw : listw (A*A)
Error: Last occurrence of "listw" must have "A" as 1st argument in
"listw (A * A)%type".

Because the conclusion of the type of constructors should be listw A in both cases.

A parametrized inductive definition can be defined using annotations instead of parameters but it will sometimes give a different (bigger) sort for the inductive definition and will produce a less convenient rule for case elimination.

See also: Sections 4.5 and 8.7.

Mutually defined inductive types

The definition of a block of mutually inductive types has the form:

\[
\text{Inductive}\ \text{ident}_1 : \text{type}_1 :=
\begin{align*}
\text{ident}_1^1 & : \text{type}_1^1 \\
\vdots \\
\text{ident}_1^n & : \text{type}_1^n \quad \text{with} \\
\quad \quad \quad \vdots \\
\text{ident}_m^m & : \text{type}_m^m.
\end{align*}
\]

It has the same semantics as the above Inductive definition for each ident$_1$, ..., ident$_m$. All names ident$_1$, ..., ident$_m$ and ident$_1^1$, ..., ident$_m^m$ are simultaneously added to the environment. Then well-typing of constructors can be checked. Each one of the ident$_1$, ..., ident$_m$ can be used on its own.

It is also possible to parametrize these inductive definitions. However, parameters correspond to a local context in which the whole set of inductive declarations is done. For this reason, the parameters must be strictly the same for each inductive types The extended syntax is:
The GALLINA specification language

Inductive ident₁ params : type₁ :=
    ident₁ : type₁
| ... |
    identₙ₁ : typeₙ₁
with
    ...
with identₘ params : typeₘ :=
    ident₁ₘ : type₁ₘ
| ... |
    identₙₘ : typeₙₘ.

Example: The typical example of a mutual inductive data type is the one for trees and forests. We assume given two types A and B as variables. It can be declared the following way,

Coq < Variables A B : Set.
Coq < Inductive tree : Set :=
Coq < node : A -> forest -> tree
Coq < with forest : Set :=
Coq < | leaf : B -> forest
Coq < | cons : tree -> forest -> forest.

This declaration generates automatically six induction principles. They are respectively called tree_rec, tree_ind, tree_rect, forest_rec, forest_ind, forest_rect. These ones are not the most general ones but are just the induction principles corresponding to each inductive part seen as a single inductive definition.

To illustrate this point on our example, we give the types of tree_rec and forest_rec.

Coq < Check tree_rec.
  tree_rec
    : forall P : tree -> Set,
      (forall (a : A) (f : forest), P (node a f)) -> forall t : tree, P t

Coq < Check forest_rec.
  forest_rec
    : forall P : forest -> Set,
      (forall b : B, P (leaf b)) ->
      (forall (t : tree) (f0 : forest), P f0 -> P (cons t f0)) ->
      forall f1 : forest, P f1

Assume we want to parametrize our mutual inductive definitions with the two type variables A and B, the declaration should be done the following way:

Coq < Inductive tree (A B:Set) : Set :=
Coq < node : A -> forest A B -> tree A B
Coq < with forest (A B:Set) : Set :=
Coq < | leaf : B -> forest A B
Coq < | cons : tree A B -> forest A B -> forest A B.

Assume we define an inductive definition inside a section. When the section is closed, the variables declared in the section and occurring free in the declaration are added as parameters to the inductive definition.

See also: Section 2.4
Co-inductive types

The objects of an inductive type are well-founded with respect to the constructors of the type. In other words, such objects contain only a finite number of constructors. Co-inductive types arise from relaxing this condition, and admitting types whose objects contain an infinity of constructors. Infinite objects are introduced by a non-ending (but effective) process of construction, defined in terms of the constructors of the type.

An example of a co-inductive type is the type of infinite sequences of natural numbers, usually called streams. It can be introduced in CoQ using the CoInductive command:

```
Coq < CoInductive Stream : Set :=
Coq <   Seq : nat -> Stream -> Stream.
Stream is defined
```

The syntax of this command is the same as the command Inductive (see Section 1.3.3). Notice that no principle of induction is derived from the definition of a co-inductive type, since such principles only make sense for inductive ones. For co-inductive ones, the only elimination principle is case analysis. For example, the usual destructors on streams `hd:Stream->nat` and `tl:Str->Str` can be defined as follows:

```
Coq < Definition hd (x:Stream) := let (a,s) := x in a.
hd is defined
Coq < Definition tl (x:Stream) := let (a,s) := x in s.
tl is defined
```

Definition of co-inductive predicates and blocks of mutually co-inductive definitions are also allowed. An example of a co-inductive predicate is the extensional equality on streams:

```
Coq < CoInductive EqSt : Stream -> Stream -> Prop :=
Coq <   eqst :
Coq <     forall s1 s2:Stream, EqSt (Seq s1) (Seq s2) -> EqSt s1 s2.
EqSt is defined
```

In order to prove the extensionally equality of two streams `s1` and `s2` we have to construct an infinite proof of equality, that is, an infinite object of type `(EqSt s1 s2)`. We will see how to introduce infinite objects in Section 1.3.4.

1.3.4 Definition of recursive functions

Definition of functions by recursion over inductive objects

This section describes the primitive form of definition by recursion over inductive objects. See Section 2.3 for more advanced constructions. The command:

```
Fixpoint ident params {struct ident } : type0 := term0
```

allows to define functions by pattern-matching over inductive objects using a fixed point construction. The meaning of this declaration is to define `ident` a recursive function with arguments specified by the binders in `params` such that `ident` applied to arguments corresponding to these binders has type `type0`, and is equivalent to the expression `term0`. The type of the `ident` is consequently `forall params , type0` and the value is equivalent to `fun params => term0`.
To be accepted, a Fixpoint definition has to satisfy some syntactical constraints on a special argument called the decreasing argument. They are needed to ensure that the Fixpoint definition always terminates. The point of the \{struct \ ident\} annotation is to let the user tell the system which argument decreases along the recursive calls. For instance, one can define the addition function as:

\[
\begin{align*}
\text{Coq} & \leftarrow \text{Fixpoint } \text{add} (n \ m: \text{nat}) \ \{\text{struct } n\} : \text{nat} := \\
& \text{Coq} \quad \text{match } n \ \text{with} \\
& \quad | \ O \Rightarrow m \\
& \quad | \ S \ p \Rightarrow S \ (\text{add} \ p \ m) \\
\end{align*}
\]

add is recursively defined (decreasing on 1st argument)

The \{struct \ ident\} annotation may be left implicit, in this case the system try successively arguments from left to right until it finds one that satisfies the decreasing condition. Note that some fixpoints may have several arguments that fit as decreasing arguments, and this choice influences the reduction of the fixpoint. Hence an explicit annotation must be used if the leftmost decreasing argument is not the desired one. Writing explicit annotations can also speed up type-checking of large mutual fixpoints.

The match operator matches a value (here \(n\)) with the various constructors of its (inductive) type. The remaining arguments give the respective values to be returned, as functions of the parameters of the corresponding constructor. Thus here when \(n\) equals \(O\) we return \(m\), and when \(n\) equals \((S \ p)\) we return \((S \ (\text{add} \ p \ m))\).

The match operator is formally described in detail in Section 4.5.4. The system recognizes that in the inductive call \((\text{add} \ p \ m)\) the first argument actually decreases because it is a pattern variable coming from \text{match} \(n\ \text{with}\).

Example: The following definition is not correct and generates an error message:

\[
\begin{align*}
\text{Coq} & \leftarrow \text{Fixpoint } \text{wrongplus} (n \ m: \text{nat}) \ \{\text{struct } n\} : \text{nat} := \\
& \text{Coq} \quad \text{match } m \ \text{with} \\
& \quad | \ O \Rightarrow n \\
& \quad | \ S \ p \Rightarrow S \ (\text{wrongplus} \ n \ p) \\
\end{align*}
\]

Coq < Coq < Error:
Recursive definition of wrongplus is ill-formed.
In environment
wrongplus : nat -> nat -> nat
\(n\) : nat
\(m\) : nat
\(p\) : nat
Recursive call to wrongplus has principal argument equal to "\(n\)"
instead of a subterm of \(n\).

because the declared decreasing argument \(n\) actually does not decrease in the recursive call. The function computing the addition over the second argument should rather be written:

\[
\begin{align*}
\text{Coq} & \leftarrow \text{Fixpoint } \text{plus} (n \ m: \text{nat}) \ \{\text{struct } m\} : \text{nat} := \\
& \text{Coq} \quad \text{match } m \ \text{with} \\
& \quad | \ O \Rightarrow n \\
& \quad | \ S \ p \Rightarrow S \ (\text{plus} \ n \ p) \\
\end{align*}
\]
The ordinary match operation on natural numbers can be mimicked in the following way.

```coq
Fixpoint nat_match
(C:Set) (f0:C) (fS:nat -> C -> C) (n:nat) {struct n} : C :=
match n with
| O => f0
| S p => fS p (nat_match C f0 fS p)
end.
```

The recursive call may not only be on direct subterms of the recursive variable \( n \) but also on a deeper subterm and we can directly write the function \( \text{mod2} \) which gives the remainder modulo 2 of a natural number.

```coq
Fixpoint mod2 (n:nat) : nat :=
match n with
| O => O
| S p => match p with
  | O => S O
  | S q => mod2 q
end
end.
```

In order to keep the strong normalization property, the fixed point reduction will only be performed when the argument in position of the decreasing argument (which type should be in an inductive definition) starts with a constructor.

The \texttt{Fixpoint} construction enjoys also the \texttt{with} extension to define functions over mutually defined inductive types or more generally any mutually recursive definitions.

**Variants:**

1. Fixpoint \( \text{ident}_1 \) \( \text{params}_1 : \text{type}_1 := \text{term}_1 \)
   with ...
   with \( \text{ident}_m \) \( \text{params}_m : \text{type}_m := \text{term}_m \)

   Allows to define simultaneously \( \text{ident}_1, \ldots, \text{ident}_m \).

**Example:** The size of trees and forests can be defined the following way:

```coq
Fixpoint tree_size (t:tree) : nat :=
match t with
| node a f => S (forest_size f)
end
with forest_size (f:forest) : nat :=
match f with
| leaf b => 1
| cons t f' => (tree_size t + forest_size f')
end.
```

A generic command \texttt{Scheme} is useful to build automatically various mutual induction principles. It is described in Section 8.14.

**Definition of recursive objects in co-inductive types**

The command:
The GALLINA specification language

introduces a method for constructing an infinite object of a coinductive type. For example, the stream
containing all natural numbers can be introduced applying the following method to the number 0 (see
Section 1.3.3 for the definition of Stream, hd and tl):

Coq < CoFixpoint from (n:nat) : Stream := Seq n (from (S n)).
from is corecursively defined

Oppositely to recursive ones, there is no decreasing argument in a co-recursive definition. To be
admissible, a method of construction must provide at least one extra constructor of the infinite object
for each iteration. A syntactical guard condition is imposed on co-recursive definitions in order to
ensure this: each recursive call in the definition must be protected by at least one constructor, and only
by constructors. That is the case in the former definition, where the single recursive call of from is
guarded by an application of Seq. On the contrary, the following recursive function does not satisfy the
guard condition:

Coq < CoFixpoint filter (p:nat -> bool) (s:Stream) : Stream :=
  if p (hd s) then Seq (hd s) (filter p (tl s)) else filter p (tl s).
Coq < Coq < Error: Recursive definition of filter is ill-formed.
In environment
  filter : (nat -> bool) -> Stream -> Stream
  p : nat -> bool
  s : Stream
Unguarded recursive call in "filter p (tl s)".

The elimination of co-recursive definition is done lazily, i.e. the definition is expanded only when it
occurs at the head of an application which is the argument of a case analysis expression. In any other
context, it is considered as a canonical expression which is completely evaluated. We can test this using
the command Eval, which computes the normal forms of a term:

Coq < Eval compute in (from 0).
  = (cofix from (n : nat) : Stream := Seq n (from (S n))) 0
    : Stream
Coq < Eval compute in (hd (from 0)).
  = 0
    : nat
Coq < Eval compute in (tl (from 0)).
  = (cofix from (n : nat) : Stream := Seq n (from (S n))) 1
    : Stream

Variants:

1. CoFixpoint ident_1 params : type_1 := term_1
   As for most constructions, arguments of co-fixpoints expressions can be introduced before the :=
sign.

2. CoFixpoint ident_1 : type_1 := term_1
   with
   ...
   with ident_m : type_m := term_m
   As in the Fixpoint command (see Section 1.3.4), it is possible to introduce a block of mutually
dependent methods.
1.3 The Vernacular

1.3.5 Statement and proofs

A statement claims a goal of which the proof is then interactively done using tactics. More on the proof editing mode, statements and proofs can be found in Chapter 7.

Theorem \text{ident} : \text{type}.

This command binds \text{type} to the name \text{ident} in the environment, provided that a proof of \text{type} is next given.

After a statement, COQ needs a proof.

Variants:

1. Lemma \text{ident} : \text{type}.
Remark \text{ident} : \text{type}.
Fact \text{ident} : \text{type}.
Corollary \text{ident} : \text{type}.
Proposition \text{ident} : \text{type}.

All these commands are synonymous of Theorem.

2. Theorem \text{ident} : \text{type} with \ldots with \text{ident} : \text{type}.

This command is useful for theorems that are proved by simultaneous induction over a mutually inductive assumption, or that state mutually dependent statements in some mutual coinductive type. It is equivalent to \text{Fixpoint} (see Section 1.3.4) or \text{CoFixpoint} (see Section 1.3.4) but using tactics to build the proof of the statements (or the body of the specification, depending on the point of view). The inductive or coinductive types on which the induction or coinduction has to be done is assumed to be non ambiguous and is guessed by the system.

Like in a Fixpoint or CoFixpoint definition, the induction hypotheses have to be used on \text{structurally smaller} arguments (for a Fixpoint) or be \text{guarded by a constructor} (for a CoFixpoint). The verification that recursive proof arguments are correct is done only at the time of registering the lemma in the environment. To know if the use of induction hypotheses is correct at some time of the interactive development of a proof, use the command \text{Guarded} (see Section 7.3.2).

The command can be used also with \text{Lemma}, \text{Remark}, etc. instead of \text{Theorem}.

3. Definition \text{ident} : \text{type}.

Allow to define a term of type \text{type} using the proof editing mode. It behaves as \text{Theorem} but is intended for the interactive definition of expression which computational behavior will be used by further commands. \text{See also: 6.9.2 and 8.5.5.}

Proof \ldots \text{Qed}.

A proof starts by the keyword \text{Proof}. Then COQ enters the proof editing mode until the proof is completed. The proof editing mode essentially contains tactics that are described in chapter 8. Besides tactics, there are commands to manage the proof editing mode. They are described in Chapter 7. When the proof is completed it should be validated and put in the environment using the keyword \text{Qed}.

Error message:
1. *ident* already exists

Remarks:

1. Several statements can be simultaneously opened.

2. Not only other statements but any vernacular command can be given within the proof editing mode. In this case, the command is understood as if it would have been given before the statements still to be proved.

3. `Proof` is recommended but can currently be omitted. On the opposite, `Qed` (or `Defined`, see below) is mandatory to validate a proof.

4. Proofs ended by `Qed` are declared opaque (see 6.9.1) and cannot be unfolded by conversion tactics (see 8.5). To be able to unfold a proof, you should end the proof by `Defined` (see below).

Variants:

1. `Proof ... Defined .`  
   Same as `Proof ... Qed .` but the proof is then declared transparent (see 6.9.2), which means it can be unfolded in conversion tactics (see 8.5).

2. `Proof ... Save .`  
   Same as `Proof ... Qed .`

3. `Goal type ... Save ident`  
   Same as `Lemma ident : type ... Save`. This is intended to be used in the interactive mode. Conversely to named lemmas, anonymous goals cannot be nested.

4. `Proof ... Admitted .`  
   Turns the current conjecture into an axiom and exits editing of current proof.
Chapter 2

Extensions of GALLINA

GALLINA is the kernel language of COQ. We describe here extensions of the Gallina’s syntax.

2.1 Record types

The Record construction is a macro allowing the definition of records as is done in many programming languages. Its syntax is described on Figure 2.1. In fact, the Record macro is more general than the usual record types, since it allows also for “manifest” expressions. In this sense, the Record construction allows to define “signatures”.

<table>
<thead>
<tr>
<th>sentence</th>
<th>++= record</th>
</tr>
</thead>
<tbody>
<tr>
<td>record</td>
<td>:= inductivity_token ident [binderlet ... binderlet] [:: sort] := [ident] { [field ; ... ; field] } .</td>
</tr>
<tr>
<td>inductivity_token</td>
<td>::= Record</td>
</tr>
<tr>
<td>field</td>
<td>::= name : type [ where notation ]</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure 2.1: Syntax for the definition of Record

In the expression

Record ident params : sort := ident_0 { ident_1 : term_1 ; ... ident_n : term_n }.

the identifier ident is the name of the defined record and sort is its type. The identifier ident_0 is the name of its constructor. If ident_0 is omitted, the default name Build_ident is used. If sort is omitted, the default sort is “Type”. The identifiers ident_1, ... ident_n are the names of fields and term_1, ... term_n their respective types. Remark that the type of ident_i may depend on the previous ident_j (for j < i). Thus the order of the fields is important. Finally, params are the parameters of the record.

More generally, a record may have explicitly defined (a.k.a. manifest) fields. For instance, Record ident [ params ] : sort := { ident_1 : type_1 ; ident_2 := term_2 ; ident_3 : type_3 } in which case the correctness of type_3 may rely on the instance term_2 of ident_2 and term_2 in turn may depend on ident_1.

Example: The set of rational numbers may be defined as:
Coq < Record Rat : Set := mkRat
Coq <  {sign : bool;
Coq <  top : nat;
Coq <  bottom : nat;
Coq <  Rat_bottom_cond : 0 <> bottom;
Coq <  Rat_irred_cond :
Coq <    forall x y z:nat, (x * y) = top \ (x * z) = bottom -> x = 1}.
Rat is defined
Rat_rect is defined
Rat_ind is defined
Rat_rec is defined
sign is defined
top is defined
bottom is defined
Rat_bottom_cond is defined
Rat_irred_cond is defined

Remark here that the field Rat_cond depends on the field bottom.

Let us now see the work done by the Record macro. First the macro generates an inductive definition with just one constructor:

Inductive ident params :sort :=
ident_0 (ident_1 : term_1) .. (ident_n : term_n).

To build an object of type ident, one should provide the constructor ident_0 with n terms filling the fields of the record.

As an example, let us define the rational 1/2:

Coq < Require Import Arith.
Coq < Theorem one_two_irred :
Coq <  forall x y z:nat, x * y = 1 \ x * z = 2 -> x = 1.
...  
Coq < Qed.
Coq < Definition half := mkRat true 1 2 (O_S 1) one_two_irred.
half is defined
Coq < Check half.
half :
    Rat

The macro generates also, when it is possible, the projection functions for destructuring an object of type ident. These projection functions have the same name that the corresponding fields. If a field is named “_” then no projection is built for it. In our example:

Coq < Eval compute in half.(top).
    = 1 : nat
Coq < Eval compute in half.(bottom).
    = 2 : nat
Coq < Eval compute in half.(Rat_bottom_cond).
    = O_S 1 : 0 <> bottom half
2.1 Record types

Variants:

1. Records declared with the keyword `Record` (or `Structure` which is equivalent to the former) cannot be recursive. However, records can be declared using the keyword `CoInductive` (resp. `Inductive`) instead, making them coinductive (resp. inductive) types.

As an example, here is how to define a type of streams over a type A as a type with a pair of destructors:

```coq
Coq < CoInductive stream (A:Type) :=
Coq <  {head : A;
Coq <  tail : stream A}.
```

Warnings:

1. Warning: `ident` cannot be defined.

It can happen that the definition of a projection is impossible. This message is followed by an explanation of this impossibility. There may be three reasons:

(a) The name `ident` already exists in the environment (see Section 1.3.1).

(b) The body of `ident` uses an incorrect elimination for `ident` (see Sections 1.3.4 and 4.5.4).

(c) The type of the projections `ident` depends on previous projections which themselves couldn’t be defined.

Error messages:

1. A record cannot be recursive

   The record name `ident` appears in the type of its fields.

2. During the definition of the one-constructor inductive definition, all the errors of inductive definitions, as described in Section 1.3.3, may also occur.

See also: Coercions and records in Section 17.9 of the chapter devoted to coercions.

Remark: `Structure` is a synonym of the keyword `Record`.

Remark: An experimental syntax for projections based on a dot notation is available. The command to activate it is

`Set Printing Projections.`

The corresponding grammar rules are given Figure 2.2. When `qualid` denotes a projection, the syntax `term . (qualid)` is equivalent to `qualid term`, the syntax `term . (qualid arg₁ ... argₙ)` to `qualid arg₁ ... argₙ term`, and the syntax `term . (@qualid term₁ ... termₙ)` to `@qualid term₁ ... termₙ term`. In each case, `term` is the object projected and the other arguments are the parameters of the inductive type.

To deactivate the printing of projections, use `Unset Printing Projections`. 
2.2 Variants and extensions of \texttt{match}

2.2.1 Multiple and nested pattern-matching

The basic version of \texttt{match} allows pattern-matching on simple patterns. As an extension, multiple nested patterns or disjunction of patterns are allowed, as in ML-like languages.

The extension just acts as a macro that is expanded during parsing into a sequence of \texttt{match} on simple patterns. Especially, a construction defined using the extended \texttt{match} is generally printed under its expanded form (see \texttt{Set Printing Matching} in section 2.2.4).

\textbf{See also:} Chapter 16.

2.2.2 Pattern-matching on boolean values: the \texttt{if} expression

For inductive types with exactly two constructors and for pattern-matchings expressions which do not depend on the arguments of the constructors, it is possible to use a \texttt{if \ldots then \ldots else \ldots} notation. For instance, the definition

\begin{verbatim}
Coq < Definition not (b:bool) :=
Coq < match b with
Coq < | true => false
Coq < | false => true
Coq < end.
not is defined
\end{verbatim}

\texttt{not} is defined can be alternatively written

\begin{verbatim}
Coq < Definition not (b:bool) := if b then false else true.
not is defined
\end{verbatim}

More generally, for an inductive type with constructors \(C_1\) and \(C_2\), we have the following equivalence:

\[
\text{match } \texttt{term \ [dep_ret_type]} \text{ with}
\]

\[
\begin{array}{c}
\text{if } \texttt{term \ [dep_ret_type]} \text{ then } \texttt{term}_1 \text{ else } \texttt{term}_2 \\
\end{array}
\equiv
\begin{array}{c}
| C_1 \ldots \_ \Rightarrow \texttt{term}_1 \\
| C_2 \ldots \_ \Rightarrow \texttt{term}_2
\end{array}
\]

\texttt{end}

Here is an example.

\begin{verbatim}
Coq < Check (fun x (H:{x=0}+{x<>0}) =>
Coq < match H with
Coq < | left _ => true
Coq < | right _ => false
Coq < end).
fun (x : nat) (H : (x = 0) + (x <> 0)) => if H then true else false
: forall x : nat, (x = 0) + (x <> 0) -> bool
\end{verbatim}

Notice that the printing uses the \texttt{if} syntax because \texttt{sumbool} is declared as such (see Section 2.2.4).
2.2 Variants and extensions of \textit{match}

2.2.3 Irrefutable patterns: the destructuring \texttt{let} variants

Pattern-matching on terms inhabiting inductive type having only one constructor can be alternatively written using \texttt{let ... in ...} constructions. There are two variants of them.

\textbf{First destructuring \texttt{let} syntax}

The expression \texttt{let \( (ident_1,\ldots,ident_n) := term_0 \) in term_1} performs case analysis on a \( term_0 \) which must be in an inductive type with one constructor having itself \( n \) arguments. Variables \( ident_1,\ldots,ident_n \) are bound to the \( n \) arguments of the constructor in expression \( term_1 \). For instance, the definition

\begin{verbatim}
Coq < Definition fst (A B:Set) (H:A * B) := match H with
Coq < | pair x y => x
Coq < end.

fst is defined
\end{verbatim}

can be alternatively written

\begin{verbatim}
Coq < Definition fst (A B:Set) (p:A * B) := let (x, _) := p in x.

fst is defined
\end{verbatim}

Notice that reduction is different from regular \texttt{let ... in ...} construction since it happens only if \( term_0 \) is in constructor form. Otherwise, the reduction is blocked.

The pretty-printing of a definition by matching on a irrefutable pattern can either be done using \texttt{match} or the \texttt{let} construction (see Section 2.2.4).

If \( term \) inhabits an inductive type with one constructor \( C \), we have an equivalence between

\begin{verbatim}
let \( (ident_1,\ldots,ident_n) \) \[dep_ret_type\] := \( term \) in \( term' \)
\end{verbatim}

and

\begin{verbatim}
match \( term \) \[dep_ret_type\] with \( C \) \( ident_1 \ldots \) \( ident_n \) =\> \( term' \) end
\end{verbatim}

\textbf{Second destructuring \texttt{let} syntax}

Another destructuring \texttt{let} syntax is available for inductive types with one constructor by giving an arbitrary pattern instead of just a tuple for all the arguments. For example, the preceding example can be written:

\begin{verbatim}
Coq < Definition fst (A B:Set) (p:A * B) := let 'pair x _ := p in x.

fst is defined
\end{verbatim}

This is useful to match deeper inside tuples and also to use notations for the pattern, as the syntax \texttt{let 'p := t in b} allows arbitrary patterns to do the deconstruction. For example:

\begin{verbatim}
Coq < let '((a,b), (c, d)) := x in (a,b,c,d).

depth_tuple is defined
\end{verbatim}

\begin{verbatim}
Coq < Notation " x 'with' p " := (exist _ x p) (at level 20).

Coq < Definition proj1_sig' (A :Set) (P : A -> Prop) (t:{ x : A | P x }) : A :=
Coq < let 'x with p := t in x.

proj1_sig' is defined
\end{verbatim}

When printing definitions which are written using this construct it takes precedence over \texttt{let} printing directives for the datatype under consideration (see Section 2.2.4).
2.2.4 Controlling pretty-printing of match expressions

The following commands give some control over the pretty-printing of match expressions.

Printing nested patterns

The Calculus of Inductive Constructions knows pattern-matching only over simple patterns. It is however convenient to re-factorize nested pattern-matching into a single pattern-matching over a nested pattern. Coq’s printer try to do such limited re-factorization.

Set Printing Matching.

This tells Coq to try to use nested patterns. This is the default behavior.

Unset Printing Matching.

This tells Coq to print only simple pattern-matching problems in the same way as the Coq kernel handles them.

Test Printing Matching.

This tells if the printing matching mode is on or off. The default is on.

Printing of wildcard pattern

Some variables in a pattern may not occur in the right-hand side of the pattern-matching clause. There are options to control the display of these variables.

Set Printing Wildcard.

The variables having no occurrences in the right-hand side of the pattern-matching clause are just printed using the wildcard symbol \_.

Unset Printing Wildcard.

The variables, even useless, are printed using their usual name. But some non dependent variables have no name. These ones are still printed using a \_.

Test Printing Wildcard.

This tells if the wildcard printing mode is on or off. The default is to print wildcard for useless variables.

Printing of the elimination predicate

In most of the cases, the type of the result of a matched term is mechanically synthesizable. Especially, if the result type does not depend of the matched term.

Set Printing Synth.

The result type is not printed when Coq knows that it can re-synthesize it.
2.2 Variants and extensions of \texttt{match}

Unset Printing Synth.
This forces the result type to be always printed.

Test Printing Synth.
This tells if the non-printing of synthesizable types is on or off. The default is to not print synthesizable types.

**Printing matching on irrefutable pattern**

If an inductive type has just one constructor, pattern-matching can be written using \texttt{let ... := ... in ...}

Add Printing Let \texttt{ident}.
This adds \texttt{ident} to the list of inductive types for which pattern-matching is written using a \texttt{let} expression.

Remove Printing Let \texttt{ident}.
This removes \texttt{ident} from this list.

Test Printing Let for \texttt{ident}.
This tells if \texttt{ident} belongs to the list.

Print Table Printing Let.
This prints the list of inductive types for which pattern-matching is written using a \texttt{let} expression.

The list of inductive types for which pattern-matching is written using a \texttt{let} expression is managed synchronously. This means that it is sensible to the command \texttt{Reset}.

**Printing matching on booleans**

If an inductive type is isomorphic to the boolean type, pattern-matching can be written using \texttt{if ... then ... else ...}

Add Printing If \texttt{ident}.
This adds \texttt{ident} to the list of inductive types for which pattern-matching is written using an \texttt{if} expression.

Remove Printing If \texttt{ident}.
This removes \texttt{ident} from this list.

Test Printing If for \texttt{ident}.
This tells if \texttt{ident} belongs to the list.

Print Table Printing If.
This prints the list of inductive types for which pattern-matching is written using an \texttt{if} expression.

The list of inductive types for which pattern-matching is written using an \texttt{if} expression is managed synchronously. This means that it is sensible to the command \texttt{Reset}.
Example

This example emphasizes what the printing options offer.

Coq < Test Printing Let for prod.
Cases on elements of prod are printed using a ‘let’ form

Coq < Print fst.
fst =
fun (A B : Set) (p : A * B) => let ‘ pair x _ := p in x
: forall A B : Set, A * B -> A
Argument scopes are {type_scope type_scope _}

Coq < Remove Printing Let prod.
Coq < Unset Printing Synth.
Coq < Unset Printing Wildcard.
Coq < Print fst.
fst =
fun (A B : Set) (p : A * B) => let ‘ pair x a := p return A in x
: forall A B : Set, A * B -> A
Argument scopes are {type_scope type_scope _}

2.3 Advanced recursive functions

The experimental command

    Function ident binder₁...binderₙ {decrease_annot} : type₀ := term₀

can be seen as a generalization of Fixpoint. It is actually a wrapper for several ways of defining a function and other useful related objects, namely: an induction principle that reflects the recursive structure of the function (see 8.7.7), and its fixpoint equality. The meaning of this declaration is to define a function ident, similarly to Fixpoint. Like in Fixpoint, the decreasing argument must be given (unless the function is not recursive), but it must not necessary be structurally decreasing. The point of the {} annotation is to name the decreasing argument and to describe which kind of decreasing criteria must be used to ensure termination of recursive calls.

The Function construction enjoys also the with extension to define mutually recursive definitions. However, this feature does not work for non structural recursive functions.

See the documentation of functional induction (see Section 8.7.7) and Functional Scheme (see Section 8.15 and 10.4) for how to use the induction principle to easily reason about the function.

Remark: To obtain the right principle, it is better to put rigid parameters of the function as first arguments. For example it is better to define plus like this:

Coq < Function plus (m n : nat) {struct n} : nat :=
Coq < match n with
Coq < | 0 => m
Coq < | S p => S (plus m p)
Coq < end.

than like this:
Coq < Function plus (n m : nat) {struct n} : nat :=
Coq < match n with
Coq < | 0 => m
Coq < | S p => S (plus p m)
Coq < end.

**Limitations**  

term$_0$ must be build as a pure pattern-matching tree (match...with) with applications only at the end of each branch. For now dependent cases are not treated.

**Error messages:**

1. The recursive argument must be specified
2. No argument name ident
3. Cannot use mutual definition with well-founded recursion or measure
4. Cannot define graph for ident... (warning)
   The generation of the graph relation (R_ident) used to compute the induction scheme of ident raised a typing error. Only the ident is defined, the induction scheme will not be generated.
   This error happens generally when:
   - the definition uses pattern matching on dependent types, which Function cannot deal with yet.
   - the definition is not a pattern-matching tree as explained above.
5. Cannot define principle(s) for ident... (warning)
   The generation of the graph relation (R_ident) succeeded but the induction principle could not be built. Only the ident is defined. Please report.
6. Cannot build functional inversion principle (warning)
   functional inversion will not be available for the function.

**See also:** 8.15, 10.4, 8.7.7

Depending on the {...} annotation, different definition mechanisms are used by Function. More precise description given below.

**Variants:**

1. Function ident binder$_1$...binder$_n$ : type$_0$ := term$_0$
   Defines the not recursive function ident as if declared with Definition. Moreover the following are defined:
   - ident_rect, ident_rec and ident_ind, which reflect the pattern matching structure of term$_0$ (see the documentation of Inductive 1.3.3);
   - The inductive R_ident corresponding to the graph of ident (silently);
   - ident_complete and ident_correct which are inversion information linking the function and its graph.
2. Function \(ident\ binder_1\ldots binder_n\ \{\text{struct} \ ident_0\} : \ type_0 := \ term_0\)

Defines the structural recursive function \(ident\) as if declared with \texttt{Fixpoint}. Moreover the following are defined:

- The same objects as above;
- The fixpoint equation of \(ident\): \(ident\_equation\).

3. Function \(ident\ binder_1\ldots binder_n\ \{\text{measure} \ term_1 \ ident_0\} : \ type_0 := \ term_0\)

4. Function \(ident\ binder_1\ldots binder_n\ \{\text{wf} \ term_1 \ ident_0\} : \ type_0 := \ term_0\)

 Defines a recursive function by well founded recursion. The \texttt{Recdef} of the standard library must be loaded for this feature. The \{\} annotation is mandatory and must be one of the following:

- \{\text{measure} \ term_1 \ ident_0\} with \(ident_0\) being the decreasing argument and \(term_1\) being a function from the type of \(ident_0\) to \(\text{nat}\) for which value on the decreasing argument decreases (for the \texttt{lt} order on \texttt{nat}) at each recursive call of \(term_0\), parameters of the function are bound in \(term_0\);
- \{\text{wf} \ term_1 \ ident_0\} with \(ident_0\) being the decreasing argument and \(term_1\) an ordering relation on the type of \(ident_0\) (i.e. of type \(T_{ident_0} \rightarrow T_{ident_0} \rightarrow \text{Prop}\)) for which the decreasing argument decreases at each recursive call of \(term_0\). The order must be well founded. Parameters of the function are bound in \(term_0\).

Depending on the annotation, the user is left with some proof obligations that will be used to define the function. These proofs are: proofs that each recursive call is actually decreasing with respect to the given criteria, and (if the criteria is \texttt{wf}) a proof that the ordering relation is well founded.

Once proof obligations are discharged, the following objects are defined:

- The same objects as with the \texttt{struct};
- The lemma \texttt{ident\_tcc} which collects all proof obligations in one property;
- The lemmas \texttt{ident\_terminate} and \texttt{ident\_F} which is needed to be inlined during extraction of \texttt{ident}.

The way this recursive function is defined is the subject of several papers by Yves Bertot and Antonia Balaa on one hand and Gilles Barthe, Julien Forest, David Pichardie and Vlad Rusu on the other hand.

	extbf{Remark:} Proof obligations are presented as several subgoals belonging to a Lemma \texttt{ident\_tcc}.

### 2.4 Section mechanism

The sectioning mechanism allows to organize a proof in structured sections. Then local declarations become available (see Section 1.3.2).
2.4.1 Section ident

This command is used to open a section named ident.

2.4.2 End ident

This command closes the section named ident. When a section is closed, all local declarations (variables and local definitions) are discharged. This means that all global objects defined in the section are generalized with respect to all variables and local definitions it depends on in the section. None of the local declarations (considered as autonomous declarations) survive the end of the section.

Here is an example:

Coq < Section s1.
Coq < Variables x y : nat.
x is assumed
y is assumed
Coq < Let y' := y.
y' is defined
Coq < Definition x' := S x.
x' is defined
Coq < Definition x'' := x' + y'.
x'' is defined
Coq < Print x'.
x' = S x
: nat
Coq < End s1.
Coq < Print x'.
x' = fun x : nat => S x
: nat -> nat
Argument scope is [nat_scope]
Coq < Print x''.
x'' = fun x y : nat => let y' := y in x' x + y'
: nat -> nat -> nat
Argument scopes are [nat_scope nat_scope]

Notice the difference between the value of x' and x'' inside section s1 and outside.

Error messages:

1. This is not the last opened section

Remarks:

1. Most commands, like Hint, Notation, option management, ... which appear inside a section are canceled when the section is closed.

2.5 Module system

The module system provides a way of packaging related elements together, as well as a mean of massive abstraction.
### 2 Extensions of GALLINA

#### 2.5.1 Module `ident`

This command is used to start an interactive module named `ident`.

**Variants:**

1. **Module** `ident module_bindings`
   
   Starts an interactive functor with parameters given by `module_bindings`.

2. **Module** `ident : module_type`
   
   Starts an interactive module specifying its module type.

3. **Module** `ident module_bindings : module_type`
   
   Starts an interactive functor with parameters given by `module_bindings`, and output module type `module_type`.

4. **Module** `ident <: module_type`
   
   Starts an interactive module satisfying `module_type`.

5. **Module** `ident module_bindings <: module_type`
   
   Starts an interactive functor with parameters given by `module_bindings`. The output module type is verified against the module type `module_type`.

6. **Module** `[Import|Export]`
   
   Behaves like `Module`, but automatically imports or exports the module.

**Reserved commands inside an interactive module:**

1. **Include** `module_expression`
   
   Includes the content of `module_expression` in the current interactive module.

2. **Include Type** `module_type`
   
   Includes the content of `module_type` in the current interactive module.

---

**Figure 2.3: Syntax of modules**

```
module_type ::= qualid
             | module_type with Definition qualid := term
             | module_type with Module qualid := qualid
             | qualid qualid ... qualid

module_binding ::= ( [Import|Export] ident ... ident : module_type )
moduleBindings ::= module_binding ... module_binding
module_expression ::= qualid ... qualid
```
2.5.2 End ident

This command closes the interactive module ident. If the module type was given the content of the module is matched against it and an error is signaled if the matching fails. If the module is basic (is not a functor) its components (constants, inductive types, submodules etc) are now available through the dot notation.

Error messages:

1. No such label ident
2. Signature components for label ident do not match
3. This is not the last opened module

2.5.3 Module ident := module_expression

This command defines the module identifier ident to be equal to module_expression.

Variants:

1. Module ident module_bindings := module_expression
   Defines a functor with parameters given by module_bindings and body module_expression.
2. Module ident module_bindings : module_type := module_expression
   Defines a functor with parameters given by module_bindings (possibly none), and output module type module_type, with body module_expression.
3. Module ident module_bindings <: module_type := module_expression
   Defines a functor with parameters given by module_bindings (possibly none) with body module_expression. The body is checked against module_type.

2.5.4 Module Type ident

This command is used to start an interactive module type ident.

Variants:

1. Module Type ident module_bindings
   Starts an interactive functor type with parameters given by module_bindings.

Reserved commands inside an interactive module type:

1. Include module_expression
   Includes the content of module_expression in the current interactive module type.
2. Include Type module_type
   Includes the content of module_type in the current interactive module type.
3. declaration_keyword Inline assum
   This declaration will be automatically unfolded at functor application.
2.5.5 End ident

This command closes the interactive module type ident.

Error messages:

1. This is not the last opened module type

2.5.6 Module Type ident := module_type

Defines a module type ident equal to module_type.

Variants:

1. Module Type ident module_bindings := module_type
   Defines a functor type ident specifying functors taking arguments module_bindings and returning module_type.

2.5.7 Declare Module ident : module_type

Declares a module ident of type module_type.

Variants:

1. Declare Module ident module_bindings : module_type
   Declares a functor with parameters module_bindings and output module type module_type.

Example

Let us define a simple module.

Coq < Module M.
Interactive Module M started
Coq < Definition T := nat.
T is defined
Coq < Definition x := 0.
x is defined
Coq < Definition y : bool.
1 subgoal

============================
bool
Coq < exact true.
Proof completed.
Coq < Defined.
exact true.
y is defined
Coq < End M.
Module M is defined

Inside a module one can define constants, prove theorems and do any other things that can be done in the toplevel. Components of a closed module can be accessed using the dot notation:
2.5 Module system

Coq < Print M.x.
M.x = 0
    : nat

A simple module type:

Coq < Module Type SIG.
Interactive Module Type SIG started
Coq < Parameter T : Set.
    T is assumed
Coq < Parameter x : T.
    x is assumed
Coq < End SIG.
Module Type SIG is defined

Inside a module type the proof editing mode is not available. Consequently commands like Definition without body, Lemma, Theorem are not allowed. In order to declare constants, use Axiom and Parameter.

Now we can create a new module from M, giving it a less precise specification: the y component is dropped as well as the body of x.

Coq < Module N : SIG with Definition T := nat := M.
Coq < Coq < Module N is defined
Coq < Print N.T.
    N.T = nat
        : Set
Coq < Print N.x.
    *** [ N.x : N.T ]
Coq < Print N.y.
    Error: N.y not a defined object.

The definition of N using the module type expression SIG with Definition T:=nat is equivalent to the following one:

Coq < Module Type SIG'.
Coq < Definition T : Set := nat.
Coq < Parameter x : T.
Coq < End SIG'.
Coq < Module N : SIG' := M.

If we just want to be sure that our implementation satisfies a given module type without restricting the interface, we can use a transparent constraint

Coq < Module P <: SIG := M.
Module P is defined
Coq < Print P.y.
    M.y = true
        : bool
Now let us create a functor, i.e. a parametric module

Coq < Module Two (X Y: SIG).
Interactive Module Two started

Coq < Definition T := (X.T * Y.T)%type.
Coq < Definition x := (X.x, Y.x).
Coq < End Two.
Module Two is defined

and apply it to our modules and do some computations

Coq < Module Q := Two M N.
Module Q is defined

Coq < Eval compute in (fst Q.x + snd Q.x).
  = N.x : nat

In the end, let us define a module type with two sub-modules, sharing some of the fields and give one of its possible implementations:

Coq < Module Type SIG2.
Interactive Module Type SIG2 started

Coq < Declare Module M1 : SIG.
Module M1 is declared

Coq < Module M2 <: SIG.
Interactive Module M2 started

Coq < Definition T := M1.T.
  T is defined

Coq < Parameter x : T.
  x is assumed

Coq < End M2.
Module M2 is defined

Coq < End SIG2.
Module Type SIG2 is defined

Coq < Module Mod <: SIG2.

Coq < Module M1.
Coq < Definition T := nat.
Coq < Definition x := 1.
Coq < End M1.
Coq < Module M2 := M.
Coq < End Mod.
Module Mod is defined

Notice that \( M \) is a correct body for the component \( M2 \) since its \( T \) component is equal \( \text{nat} \) and hence \( M1.T \) as specified.

Remarks:
1. Modules and module types can be nested components of each other.

2. When a module declaration is started inside a module type, the proof editing mode is still unavailable.

3. One can have sections inside a module or a module type, but not a module or a module type inside a section.

4. Commands like Hint or Notation can also appear inside modules and module types. Note that in case of a module definition like:

   Module N : SIG := M.
   or
   Module N : SIG.
   ... 
   End N.

   hints and the like valid for N are not those defined in M (or the module body) but the ones defined in SIG.

2.5.8 Import qualid

If qualid denotes a valid basic module (i.e. its module type is a signature), makes its components available by their short names.

   Example:

   Coq < Module Mod.
   Interactive Module Mod started
   Coq < Definition T:=nat.
   T is defined
   Coq < Check T.
   T
   : Set
   Coq < End Mod.
   Module Mod is defined
   Coq < Check Mod.T.
   Mod.T
   : Set
   Coq < Check T. (* Incorrect ! *)
   Toplevel input, characters 6-7:
   > Check T.
   > ^
   Error: The reference T was not found in the current environment.
   Coq < Import Mod.
   Coq < Check T. (* Now correct *)
   T
   : Set
Some features defined in modules are activated only when a module is imported. This is for instance the case of notations (see Section 12.1).

**Variants:**

1. **Export qualid**
   
   When the module containing the command `Export qualid` is imported, `qualid` is imported as well.

**Error messages:**

1. `qualid` is not a module

**Warnings:**

1. Warning: Trying to mask the absolute name `qualid`!

### 2.5.9 Print Module `ident`

Prints the module type and (optionally) the body of the module `ident`.

### 2.5.10 Print Module Type `ident`

Prints the module type corresponding to `ident`.

### 2.5.11 Locate Module `qualid`

Prints the full name of the module `qualid`.

### 2.6 Libraries and qualified names

#### 2.6.1 Names of libraries and files

**Libraries**

The theories developed in Coq are stored in library files which are hierarchically classified into libraries and sublibraries. To express this hierarchy, library names are represented by qualified identifiers `qualid`, i.e. as list of identifiers separated by dots (see Section 1.2.3). For instance, the library file `Mult` of the standard Coq library Arith has name `Coq.Arith.Mult`. The identifier that starts the name of a library is called a library root. All library files of the standard library of Coq have reserved root `Coq` but library file names based on other roots can be obtained by using `coqc` options `-I` or `-R` (see Section 13.5). Also, when an interactive Coq session starts, a library of root `Top` is started, unless option `-top` or `-notop` is set (see Section 13.5).

As library files are stored on the file system of the underlying operating system, a translation from file-system names to Coq names is needed. In this translation, names in the file system are called physical paths while Coq names are contrastingly called logical names. Logical names are mapped to physical paths using the commands `Add LoadPath` or `Add Rec LoadPath` (see Sections 6.5.3 and 6.5.4).
2.6 Libraries and qualified names

2.6.2 Qualified names

Library files are modules which possibly contain submodules which eventually contain constructions (axioms, parameters, definitions, lemmas, theorems, remarks or facts). The absolute name, or full name, of a construction in some library file is a qualified identifier starting with the logical name of the library file, followed by the sequence of submodules names encapsulating the construction and ended by the proper name of the construction. Typically, the absolute name `Coq.Init.Logic.eq` denotes Leibniz’ equality defined in the module `Logic` in the sublibrary `Init` of the standard library of COQ.

The proper name that ends the name of a construction is the short name (or sometimes base name) of the construction (for instance, the short name of `Coq.Init.Logic.eq` is `eq`). Any partial suffix of the absolute name is a partially qualified name (e.g. `Logic.eq` is a partially qualified name for `Coq.Init.Logic.eq`). Especially, the short name of a construction is its shortest partially qualified name.

COQ does not accept two constructions (definition, theorem, ...) with the same absolute name but different constructions can have the same short name (or even same partially qualified names as soon as the full names are different).

Notice that the notion of absolute, partially qualified and short names also applies to library file names.

Visibility

COQ maintains a table called name table which maps partially qualified names of constructions to absolute names. This table is updated by the commands Require (see 6.4.1), Import and Export (see 2.5.8) and also each time a new declaration is added to the context. An absolute name is called visible from a given short or partially qualified name when this latter name is enough to denote it. This means that the short or partially qualified name is mapped to the absolute name in COQ name table.

A similar table exists for library file names. It is updated by the vernacular commands Add LoadPath and Add Rec LoadPath (or their equivalent as options of the COQ executables, -I and -R).

It may happen that a visible name is hidden by the short name or a qualified name of another construction. In this case, the name that has been hidden must be referred to using one more level of qualification. To ensure that a construction always remains accessible, absolute names can never be hidden.

Examples:

```coq
Coq < Check 0.
0 : nat

Coq < Definition nat := bool.
nat is defined

Coq < Check 0.
0 : Datatypes.nat

Coq < Check Datatypes.nat.
Datatypes.nat : Set

Coq < Locate nat.
Constant Top.nat
Inductive Coq.Init.Datatypes.nat
```
(shorter name to refer to it in current context is Datatypes.nat)

See also: Command Locate in Section 6.2.10 and Locate Library in Section 6.5.11.

### 2.7 Implicit arguments

An implicit argument of a function is an argument which can be inferred from contextual knowledge. There are different kinds of implicit arguments that can be considered implicit in different ways. There are also various commands to control the setting or the inference of implicit arguments.

#### 2.7.1 The different kinds of implicit arguments

**Implicit arguments inferable from the knowledge of other arguments of a function**

The first kind of implicit arguments covers the arguments that are inferable from the knowledge of the type of other arguments of the function, or of the type of the surrounding context of the application. Especially, such implicit arguments correspond to parameters dependent in the type of the function. Typical implicit arguments are the type arguments in polymorphic functions. There are several kinds of such implicit arguments.

**Strict Implicit Arguments.** An implicit argument can be either strict or non strict. An implicit argument is said *strict* if, whatever the other arguments of the function are, it is still inferable from the type of some other argument. Technically, an implicit argument is strict if it corresponds to a parameter which is not applied to a variable which itself is another parameter of the function (since this parameter may erase its arguments), not in the body of a match, and not itself applied or matched against patterns (since the original form of the argument can be lost by reduction).

For instance, the first argument of

```coq
cons : forall A:Set, A -> list A -> list A
```

in module `List.v` is strict because `list` is an inductive type and `A` will always be inferable from the type `list A` of the third argument of `cons`. On the contrary, the second argument of a term of type

```coq
forall P:nat->Prop, forall n:nat, P n -> ex nat P
```

is implicit but not strict, since it can only be inferred from the type `P n` of the third argument and if `P` is, e.g., `fun _ => True`, it reduces to an expression where `n` does not occur any longer. The first argument `P` is implicit but not strict either because it can only be inferred from `P n` and `P` is not canonically inferable from an arbitrary `n` and the normal form of `P n` (consider e.g. that `n` is `0` and the third argument has type `True`, then any `P` of the form `fun n => match n with 0 => True | _ => anything` end would be a solution of the inference problem).

**Contextual Implicit Arguments.** An implicit argument can be *contextual* or not. An implicit argument is said *contextual* if it can be inferred only from the knowledge of the type of the context of the current expression. For instance, the only argument of

```coq
nil : forall A:Set, list A
```

is contextual. Similarly, both arguments of a term of type

```coq
forall P:nat->Prop, forall n:nat, P n \&\& n = 0
```

are contextual (moreover, `n` is strict and `P` is not).
2.7 Implicit arguments

**Reversible-Pattern Implicit Arguments.** There is another class of implicit arguments that can be reinferred unambiguously if all the types of the remaining arguments are known. This is the class of implicit arguments occurring in the type of another argument in position of reversible pattern, which means it is at the head of an application but applied only to uninstantiated distinct variables. Such an implicit argument is called *reversible-pattern implicit argument*. A typical example is the argument $P$ of `nat_rec` in

```coq
nat_rec : forall P : nat -> Set, P 0 -> (forall n : nat, P n -> P (S n)) -> forall x : nat, P x.
```

($P$ is reinferable by abstracting over $n$ in the type $P n$).

See Section 2.7.9 for the automatic declaration of reversible-pattern implicit arguments.

**Implicit arguments inferable by resolution**

This corresponds to a class of non dependent implicit arguments that are solved based on the structure of their type only.

### 2.7.2 Maximal or non maximal insertion of implicit arguments

In case a function is partially applied, and the next argument to be applied is an implicit argument, two disciplines are applicable. In the first case, the function is considered to have no arguments furtherly: one says that the implicit argument is not maximally inserted. In the second case, the function is considered to be implicitly applied to the implicit arguments it is waiting for: one says that the implicit argument is maximally inserted.

Each implicit argument can be declared to have to be inserted maximally or non maximally. This can be governed argument per argument by the command `Implicit Arguments` (see 2.7.4) or globally by the command `Set Maximal Implicit Insertion` (see 2.7.10). See also Section 2.7.12.

### 2.7.3 Casual use of implicit arguments

In a given expression, if it is clear that some argument of a function can be inferred from the type of the other arguments, the user can force the given argument to be guessed by replacing it by “$_$”. If possible, the correct argument will be automatically generated.

**Error messages:**

1. Cannot infer a term for this placeholder

   COQ was not able to deduce an instantiation of a “$_$”.

### 2.7.4 Declaration of implicit arguments for a constant

In case one wants that some arguments of a given object (constant, inductive types, constructors, assumptions, local or not) are always inferred by Coq, one may declare once and for all which are the expected implicit arguments of this object. There are two ways to do this, a-priori and a-posteriori.
Implicit Argument Binders

In the first setting, one wants to explicitly give the implicit arguments of a constant as part of its definition. To do this, one has to surround the bindings of implicit arguments by curly braces:

\[
\text{Coq < Definition id } \{A : \text{Type}\} \ (x : A) \ : A := x.
\]

\textit{id is defined}

This automatically declares the argument \(A\) of \(\text{id}\) as a maximally inserted implicit argument. One can then do as-if the argument was absent in every situation but still be able to specify it if needed:

\[
\begin{align*}
\text{Coq < Definition compose } \langle A B C \rangle \ (g : B \to C) \ (f : A \to B) & := \\
\text{Coq < fun x => g (f x).} \\
\text{compose is defined}
\end{align*}
\]

\[
\text{Coq < Goal forall A, compose id id = id } \langle A := A \rangle.
\]

1 subgoal

\[
\begin{align*}
\text{forall A : Type, compose id id = id} \\
\text{-----------}
\end{align*}
\]

The syntax is supported in all top-level definitions: \texttt{Definition}, \texttt{Fixpoint}, \texttt{Lemma} and so on. For (co-)inductive datatype declarations, the semantics are the following: an inductive parameter declared as an implicit argument need not be repeated in the inductive definition but will become implicit for the constructors of the inductive only, not the inductive type itself. For example:

\[
\begin{align*}
\text{Coq < Inductive list } \langle A : \text{Type} \rangle & : \text{Type} := \\
\text{Coq < | nil : list} \\
\text{Coq < | cons : A -> list -> list.} \\
\text{list is defined} \\
\text{list_rect is defined} \\
\text{list_ind is defined} \\
\text{list_rec is defined}
\end{align*}
\]

\[
\text{Coq < Print list.} \\
\text{Inductive list } \langle A : \text{Type} \rangle : \text{Type} := \text{nil : list | cons : A -> list -> list} \\
\text{For list: Argument A is implicit and maximally inserted} \\
\text{For nil: Argument A is implicit and maximally inserted} \\
\text{For cons: Argument A is implicit and maximally inserted} \\
\text{For list: Argument scope is [type_scope]} \\
\text{For nil: Argument scope is [type_scope]} \\
\text{For cons: Argument scopes are [type_scope _ _]}
\]

One can always specify the parameter if it is not uniform using the usual implicit arguments disambiguation syntax.

The Implicit Arguments Vernacular Command

To set implicit arguments for a constant a-posteriori, one can use the command:

\[
\text{Implicit Arguments qualid [ possibly_bracketed_ident \ldots possibly_bracketed_ident ]}
\]
where the list of `possibly_bracketed_ident` is the list of parameters to be declared implicit, each of the identifier of the list being optionally surrounded by square brackets, then meaning that this parameter has to be maximally inserted.

After the above declaration is issued, implicit arguments can just (and have to) be skipped in any expression involving an application of `qualid`.

**Variants:**

1. **Global Implicit Arguments**
   
   `qualid [ possibly_bracketed_ident ... possibly_bracketed_ident ]`

   Tells to recompute the implicit arguments of `qualid` after ending of the current section if any, enforcing the implicit arguments known from inside the section to be the ones declared by the command.

2. **Local Implicit Arguments**

   `qualid [ possibly_bracketed_ident ... possibly_bracketed_ident ]`

   When in a module, tells not to activate the implicit arguments of `qualid` declared by this commands to contexts that requires the module.

**Example:**

```coq
Coq < Inductive list (A:Type) : Type :=
Coq <   | nil : list A
Coq <   | cons : A -> list A -> list A.
Coq < Check (cons nat 3 (nil nat)).
Coq < cons nat 3 (nil nat)
Coq < : list nat
Coq < Implicit Arguments cons [A].
Coq < Implicit Arguments nil [A].
Coq < Check (cons 3 nil).
Coq < cons 3 nil
Coq < : list nat
Coq < Fixpoint map (A B:Type) (f:A->B) (l:list A) : list B :=
Coq <   match l with nil => nil | cons a t => cons (f a) (map A B f t) end.
Coq < map is recursively defined (decreasing on 4th argument)
Coq < Fixpoint length (A:Type) (l:list A) : nat :=
Coq <   match l with nil => 0 | cons _ m => S (length A m) end.
Coq < length is recursively defined (decreasing on 2nd argument)
Coq < Implicit Arguments map [A B].
Coq < Implicit Arguments length [[A]]. (* A has to be maximally inserted *)
Coq < Check (fun l:list (list nat) => map length l).
Coq < fun l : list (list nat) => map length l
Coq < : list (list nat) -> list nat
```

**Remark:** To know which are the implicit arguments of an object, use the command `Print Implicit` (see 2.7.12).

**Remark:** If the list of arguments is empty, the command removes the implicit arguments of `qualid`.
2.7.5 Automatic declaration of implicit arguments for a constant

Coq can also automatically detect what are the implicit arguments of a defined object. The command is just

```
Implicit Arguments qualid
```

The auto-detection is governed by options telling if strict, contextual, or reversible-pattern implicit arguments must be considered or not (see Sections 2.7.7, 2.7.8, 2.7.9 and also 2.7.10).

Variants:

1. Global Implicit Arguments `qualid`
   - Tells to recompute the implicit arguments of `qualid` after ending of the current section if any.

2. Local Implicit Arguments `qualid`
   - When in a module, tells not to activate the implicit arguments of `qualid` computed by this declaration to contexts that requires the module.

Example:

```
Coq < Inductive list (A:Set) : Set :=
Coq <   | nil : list A
Coq <   | cons : A -> list A -> list A.
Coq < Implicit Arguments cons.
Coq < Print Implicit cons.
cons : forall A : Set, A -> list A -> list A
Argument A is implicit
Coq < Implicit Arguments nil.
Coq < Print Implicit nil.
nil : forall A : Set, list A
No implicit arguments
Coq < Set Contextual Implicit.
Coq < Implicit Arguments nil.
Coq < Print Implicit nil.
nil : forall A : Set, list A
Argument A is implicit and maximally inserted
```

The computation of implicit arguments takes account of the unfolding of constants. For instance, the variable `p` below has type `(Transitivity R)` which is reducible to `forall x,y:U, R x y -> forall z:U, R y z -> R x z`. As the variables `x`, `y` and `z` appear strictly in body of the type, they are implicit.

```
Coq < Variable X : Type.
Coq < Definition Relation := X -> X -> Prop.
Coq < Definition Transitivity (R:Relation) :=
Coq <   forall x y:X, R x y -> forall z:X, R y z -> R x z.
Coq < Variables (R : Relation) (p : Transitivity R).
```
2.7 Implicit arguments

Coq < Implicit Arguments p.

Coq < Print p.

Coq < Variables (a b c : X) (r1 : R a b) (r2 : R b c).

Coq < Check (p r1 r2).

2.7.6 Mode for automatic declaration of implicit arguments

In case one wants to systematically declare implicit the arguments detectable as such, one may switch to the automatic declaration of implicit arguments mode by using the command

    Set Implicit Arguments.

Conversely, one may unset the mode by using Unset Implicit Arguments. The mode is off by default. Auto-detection of implicit arguments is governed by options controlling whether strict and contextual implicit arguments have to be considered or not.

2.7.7 Controlling strict implicit arguments

When the mode for automatic declaration of implicit arguments is on, the default is to automatically set implicit only the strict implicit arguments plus, for historical reasons, a small subset of the non strict implicit arguments. To relax this constraint and to set implicit all non strict implicit arguments by default, use the command

    Unset Strict Implicit.

Conversely, use the command Set Strict Implicit to restore the original mode that declares implicit only the strict implicit arguments plus a small subset of the non strict implicit arguments.

In the other way round, to capture exactly the strict implicit arguments and no more than the strict implicit arguments, use the command:

    Set Strongly Strict Implicit.

Conversely, use the command Unset Strongly Strict Implicit to let the option “Strict Implicit” decide what to do.

Remark: In versions of Coq prior to version 8.0, the default was to declare the strict implicit arguments as implicit.
2 Extensions of GALINA

<table>
<thead>
<tr>
<th>term</th>
<th>+++=</th>
<th>@ qualid term</th>
<th>...</th>
<th>term</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>@ qualid</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>qualid argument</td>
<td>...</td>
<td>argument</td>
</tr>
</tbody>
</table>

Figure 2.4: Syntax for explicitly giving implicit arguments

2.7.8 Controlling contextual implicit arguments

By default, COQ does not automatically set implicit the contextual implicit arguments. To tell COQ to infer also contextual implicit argument, use command

```
Set Contextual Implicit.
```

Conversely, use command `Unset Contextual Implicit` to unset the contextual implicit mode.

2.7.9 Controlling reversible-pattern implicit arguments

By default, COQ does not automatically set implicit the reversible-pattern implicit arguments. To tell COQ to infer also reversible-pattern implicit argument, use command

```
Set Reversible Pattern Implicit.
```

Conversely, use command `Unset Reversible Pattern Implicit` to unset the reversible-pattern implicit mode.

2.7.10 Controlling the insertion of implicit arguments not followed by explicit arguments

Implicit arguments can be declared to be automatically inserted when a function is partially applied and the next argument of the function is an implicit one. In case the implicit arguments are automatically declared (with the command `Set Implicit Arguments`), the command

```
Set Maximal Implicit Insertion.
```

is used to tell to declare the implicit arguments with a maximal insertion status. By default, automatically declared implicit arguments are not declared to be insertable maximally. To restore the default mode for maximal insertion, use command `Unset Maximal Implicit Insertion`.

2.7.11 Explicit applications

In presence of non strict or contextual argument, or in presence of partial applications, the synthesis of implicit arguments may fail, so one may have to give explicitly certain implicit arguments of an application. The syntax for this is `(ident := term)` where `ident` is the name of the implicit argument and `term` is its corresponding explicit term. Alternatively, one can locally deactivate the hiding of implicit arguments of a function by using the notation `@qualid term1 . . termn`. This syntax extension is given Figure 2.4.

Example (continued):
2.7 Implicit arguments

Coq < Check (p r1 (z:=c)).
p r1 (z:=c)
    : R b c -> R a c
Coq < Check (p (x:=a) (y:=b) r1 (z:=c) r2).
p r1 r2
    : R a c

2.7.12 Displaying what the implicit arguments are

To display the implicit arguments associated to an object, and to know if each of them is to be used maximally or not, use the command

    Print Implicit qualid.

2.7.13 Explicit displaying of implicit arguments for pretty-printing

By default the basic pretty-printing rules hide the inferable implicit arguments of an application. To force printing all implicit arguments, use command

    Set Printing Implicit.

Conversely, to restore the hiding of implicit arguments, use command

    Unset Printing Implicit.

    By default the basic pretty-printing rules display the implicit arguments that are not detected as strict implicit arguments. This “defensive” mode can quickly make the display cumbersome so this can be deactivated by using the command

    Unset Printing Implicit Defensive.

Conversely, to force the display of non strict arguments, use command

    Set Printing Implicit Defensive.

See also: Set Printing All in Section 2.9.

2.7.14 Interaction with subtyping

When an implicit argument can be inferred from the type of more than one of the other arguments, then only the type of the first of these arguments is taken into account, and not an upper type of all of them. As a consequence, the inference of the implicit argument of “=” fails in

Coq < Check nat = Prop.

    but succeeds in

Coq < Check Prop = nat.


2.7.15 Canonical structures

A canonical structure is an instance of a record/structure type that can be used to solve equations involving implicit arguments. Assume that qualid denotes an object \((\text{Build\_struc}\ c_1 \ldots c_n)\) in the structure \text{struct} of which the fields are \(x_1, \ldots, x_n\). Assume that \text{qualid} is declared as a canonical structure using the command

```plaintext
Canonical Structure qualid.
```

Then, each time an equation of the form \((x_i \_\_) = \beta\delta\varepsilon c_i\) has to be solved during the type-checking process, \text{qualid} is used as a solution. Otherwise said, \text{qualid} is canonically used to extend the field \(c_i\) into a complete structure built on \(c_i\).

Canonical structures are particularly useful when mixed with coercions and strict implicit arguments. Here is an example.

```plaintext
Coq < Require Import Relations.
Coq < Require Import EqNat.
Coq < Set Implicit Arguments.
Coq < Unset Strict Implicit.
Coq < Structure Setoid : Type :=
  {Carrier :> Set;
   Equal : relation Carrier;
   Prf_equiv : equivalence Carrier Equal}.
Coq < Definition is_law (A B:Setoid) (f:A -> B) :=
  forall x y:A, Equal x y -> Equal (f x) (f y).
Coq < Axiom eq_nat_equiv : equivalence nat eq_nat.
Coq < Definition nat_setoid : Setoid := Build_Setoid eq_nat_equiv.
Coq < Canonical Structure nat_setoid.
```

Thanks to \text{nat\_setoid} declared as canonical, the implicit arguments \(A\) and \(B\) can be synthesized in the next statement.

```plaintext
Coq < Lemma is_law_S : is_law S.
1 subgoal

===========================
is_law (A:=nat_setoid) (B:=nat_setoid) S
```

Remark: If a same field occurs in several canonical structure, then only the structure declared first as canonical is considered.

Variants:

1. Canonical Structure \text{ident} := \text{term} : \text{type}.
   Canonical Structure \text{ident} := \text{term}.
   Canonical Structure \text{ident} : \text{type} := \text{term}.

   These are equivalent to a regular definition of \text{ident} followed by the declaration

   ```plaintext
   Canonical Structure ident.
   ```

See also: more examples in user contribution category (Rocq/ALGEBRA).
Print Canonical Projections.

This displays the list of global names that are components of some canonical structure. For each of them, the canonical structure of which it is a projection is indicated. For instance, the above example gives the following output:

```
Coq < Print Canonical Projections.
eq_nat_equiv <- Prf_equiv ( nat_setoid )
eq_nat <- Equal ( nat_setoid )
nat <- Carrier ( nat_setoid )
```

2.7.16 Implicit types of variables

It is possible to bind variable names to a given type (e.g. in a development using arithmetic, it may be convenient to bind the names $n$ or $m$ to the type $\text{nat}$ of natural numbers). The command for that is

```
Implicit Types ident ... ident : type
```

The effect of the command is to automatically set the type of bound variables starting with `ident` (either `ident` itself or `ident` followed by one or more single quotes, underscore or digits) to be `type` (unless the bound variable is already declared with an explicit type in which case, this latter type is considered).

Example:

```
Coq < Require Import List.
Coq < Implicit Types m n : nat.
Coq < Lemma cons_inj_nat : forall m n l, n :: l = m :: l -> n = m.
  1 subgoal
    ==============
    forall m n (l : list nat), n :: l = m :: l -> n = m
Coq < intros m n.
  1 subgoal
    m : nat
    n : nat
    ==============
    forall l : list nat, n :: l = m :: l -> n = m
Coq < Lemma cons_inj_bool : forall (m n:bool) l, n :: l = m :: l -> n = m.
  1 subgoal
    ============== 
    forall (m n : bool) (l : list bool), n :: l = m :: l -> n = m
```

Variants:

1. Implicit Type `ident` : `type`
   This is useful for declaring the implicit type of a single variable.
2.8 Coercions

Coercions can be used to implicitly inject terms from one class in which they reside into another one. A class is either a sort (denoted by the keyword `Sortclass`), a product type (denoted by the keyword `Funclass`), or a type constructor (denoted by its name), e.g. an inductive type or any constant with a type of the form `forall (x_1 : A_1) .. (x_n : A_n), s` where `s` is a sort.

Then the user is able to apply an object that is not a function, but can be coerced to a function, and more generally to consider that a term of type A is of type B provided that there is a declared coercion between A and B. The main command is

```
Coercion qualid : class1 >-> class2.
```

which declares the construction denoted by `qualid` as a coercion between `class1` and `class2`.

More details and examples, and a description of the commands related to coercions are provided in Chapter 17.

2.9 Printing constructions in full

Coercions, implicit arguments, the type of pattern-matching, but also notations (see Chapter 12) can obfuscate the behavior of some tactics (typically the tactics applying to occurrences of subterms are sensitive to the implicit arguments). The command

```
Set Printing All.
```

deactivates all high-level printing features such as coercions, implicit arguments, returned type of pattern-matching, notations and various syntactic sugar for pattern-matching or record projections. Otherwise said, `Set Printing All` includes the effects of the commands `Set Printing Implicit, Set Printing Coercions, Set Printing Synth, Unset Printing Projections and Unset Printing Notations`. To reactivate the high-level printing features, use the command

```
Unset Printing All.
```

2.10 Printing universes

The following command:

```
Set Printing Universes
```

activates the display of the actual level of each occurrence of `Type`. See Section 4.1.1 for details. This wizard option, in combination with `Set Printing All` (see section 2.9) can help to diagnose failures to unify terms apparently identical but internally different in the Calculus of Inductive Constructions. To reactivate the display of the actual level of the occurrences of `Type`, use

```
Unset Printing Universes.
```

The constraints on the internal level of the occurrences of `Type` (see Section 4.1.1) can be printed using the command

```
Print Universes.
```
Chapter 3

The COQ library

The COQ library is structured into two parts:

The initial library: it contains elementary logical notions and data-types. It constitutes the basic state of the system directly available when running COQ;

The standard library: general-purpose libraries containing various developments of COQ axiomatizations about sets, lists, sorting, arithmetic, etc. This library comes with the system and its modules are directly accessible through the Require command (see Section 6.4.1);

In addition, user-provided libraries or developments are provided by COQ users’ community. These libraries and developments are available for download at http://coq.inria.fr (see Section 3.3). The chapter briefly reviews the COQ libraries.

3.1 The basic library

This section lists the basic notions and results which are directly available in the standard COQ system\(^1\).

3.1.1 Notations

This module defines the parsing and pretty-printing of many symbols (infixes, prefixes, etc.). However, it does not assign a meaning to these notations. The purpose of this is to define and fix once for all the precedence and associativity of very common notations. The main notations fixed in the initial state are listed on Figure 3.1.

3.1.2 Logic

The basic library of COQ comes with the definitions of standard (intuitionistic) logical connectives (they are defined as inductive constructions). They are equipped with an appealing syntax enriching the (subclass form) of the syntactic class term. The syntax extension is shown on Figure 3.2.

Remark: Implication is not defined but primitive (it is a non-dependent product of a proposition over another proposition). There is also a primitive universal quantification (it is a dependent product over a

\(^1\)Most of these constructions are defined in the Prelude module in directory theories/Init at the COQ root directory; this includes the modules Notations, Logic, Datatypes, Specif, Peano, Wf and Tactics. Module Logic_Type also makes it in the initial state
### 3 The Coq library

<table>
<thead>
<tr>
<th>Notation</th>
<th>Precedence</th>
<th>Associativity</th>
</tr>
</thead>
<tbody>
<tr>
<td>_ &lt;-&gt; _</td>
<td>95</td>
<td>no</td>
</tr>
<tr>
<td>_ / _</td>
<td>85</td>
<td>right</td>
</tr>
<tr>
<td>_ (_)</td>
<td>80</td>
<td>right</td>
</tr>
<tr>
<td>_ = _</td>
<td>75</td>
<td>right</td>
</tr>
<tr>
<td>_ = _ = _</td>
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<td>no</td>
</tr>
<tr>
<td>_ = _ :&gt;_ _</td>
<td>70</td>
<td>no</td>
</tr>
<tr>
<td>_ &lt;&gt; _ :&gt;_ _</td>
<td>70</td>
<td>no</td>
</tr>
<tr>
<td>_ &lt;&gt; _</td>
<td>70</td>
<td>no</td>
</tr>
<tr>
<td>_ &lt; _</td>
<td>70</td>
<td>no</td>
</tr>
<tr>
<td>_ &gt; _</td>
<td>70</td>
<td>no</td>
</tr>
<tr>
<td>_ &lt;= _</td>
<td>70</td>
<td>no</td>
</tr>
<tr>
<td>_ &gt;= _</td>
<td>70</td>
<td>no</td>
</tr>
<tr>
<td>_ &lt; _ &lt; _</td>
<td>70</td>
<td>no</td>
</tr>
<tr>
<td>_ &lt; _ &lt;= _</td>
<td>70</td>
<td>no</td>
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<td>no</td>
</tr>
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<td>_ &lt;= _ &lt;= _</td>
<td>70</td>
<td>no</td>
</tr>
<tr>
<td>_ + _</td>
<td>50</td>
<td>left</td>
</tr>
<tr>
<td>_</td>
<td></td>
<td>_</td>
</tr>
<tr>
<td>_ - _</td>
<td>50</td>
<td>left</td>
</tr>
<tr>
<td>_ * _</td>
<td>40</td>
<td>left</td>
</tr>
<tr>
<td>_ &amp; &amp; _</td>
<td>40</td>
<td>left</td>
</tr>
<tr>
<td>_ / _</td>
<td>40</td>
<td>left</td>
</tr>
<tr>
<td>_ ^ _</td>
<td>30</td>
<td>right</td>
</tr>
</tbody>
</table>

Figure 3.1: Notations in the initial state

```
form ::= True                  (True)
       False                 (False)
       ~ form           (not)
       form \(\_\) form    (and)
       form \(\_\)/ form    (or)
       form -> form        (primitive implication)
       form <-> form        (iff)
       forall ident : type , form  (primitive for all)
       exists ident [: specif], form  (ex)
       exists2 ident [: specif], form & form  (ex2)
       term = term             (eq)
       term = term :> specif    (eq)
```

Figure 3.2: Syntax of formulas
3.1 The basic library

The primitive universal quantification allows both first-order and higher-order quantification.

**Propositional Connectives**

First, we find propositional calculus connectives:

Coq < Inductive True : Prop := I.
Coq < Inductive False : Prop := .
Coq < Definition not (A: Prop) := A -> False.
Coq < Inductive and (A B:Prop) : Prop := conj (_:A) (_:B).
Coq < Section Projections.
Coq < Variables A B : Prop.
Coq < Theorem proj1 : A \& B -> A.
Coq < Theorem proj2 : A \& B -> B.
Coq < End Projections.

Coq < Inductive or (A B:Prop) : Prop :=
Coq < | or_introl (_:A)
Coq < | or_intror (_:B).
Coq < Definition iff (P Q:Prop) := (P -> Q) \&\& (Q -> P).
Coq < Definition IF_then_else (P Q R:Prop) := P \&\& Q \&\& ~ P \&\& R.

**Quantifiers**

Then we find first-order quantifiers:

Coq < Definition all (A:Set) (P:A -> Prop) := forall x:A, P x.
Coq < Inductive ex (A: Set) (P:A -> Prop) : Prop :=
Coq < ex_intro (x:A) (_:P x).
Coq < Inductive ex2 (A:Set) (P Q:A -> Prop) : Prop :=
Coq < ex_intro2 (x:A) (_:P x) (_:Q x).

The following abbreviations are allowed:

| exists x:A, P | ex A (fun x:A => P) |
| exists x, P   | ex _ (fun x => P)   |
| exists2 x:A, P & Q | ex2 A (fun x:A => P) (fun x:A => Q) |
| exists2 x, P & Q | ex2 _ (fun x => P) (fun x => Q) |

The type annotation “:A” can be omitted when A can be synthesized by the system.
Equality

Then, we find equality, defined as an inductive relation. That is, given a type \( A \) and an \( x \) of type \( A \),
the predicate \( \text{eq } A \ x \) is the smallest one which contains \( x \). This definition, due to Christine Paulin-Mohring,
is equivalent to define \( \text{eq} \) as the smallest reflexive relation, and it is also equivalent to Leibniz’ equality.

\[
\text{Coq} < \text{Inductive eq (A:Type) (x:A) : A -> Prop :=}
\text{Coq} < \quad \text{refl_equal : eq A x x.}
\]

Lemmas

Finally, a few easy lemmas are provided.

\[
\text{Coq} < \text{Theorem absurd : forall A C:Prop, A -> ~ A -> C.}
\]

\[
\text{Coq} < \text{Section equality.}
\]

\[
\text{Coq} < \text{Variables A B : Type.}
\]

\[
\text{Coq} < \text{Variable f : A -> B.}
\]

\[
\text{Coq} < \text{Variables x y z : A.}
\]

\[
\text{Coq} < \text{Theorem sym_eq : x = y -> y = x.}
\]

\[
\text{Coq} < \text{Theorem trans_eq : x = y -> y = z -> x = z.}
\]

\[
\text{Coq} < \text{Theorem f_equal : x = y -> f x = f y.}
\]

\[
\text{Coq} < \text{Theorem sym_not_eq : x <> y -> y <> x.}
\]

\[
\text{Coq} < \text{End equality.}
\]

\[
\text{Coq} < \text{Definition eq_ind_r :}
\]

\[
\text{Coq} < \quad \text{forall (A:Type) (x:A) (P:A -> Prop), P x -> forall y:A, y = x -> P y.}
\]

\[
\text{Coq} < \text{Definition eq_rec_r :}
\]

\[
\text{Coq} < \quad \text{forall (A:Type) (x:A) (P:A -> Set), P x -> forall y:A, y = x -> P y.}
\]

\[
\text{Coq} < \text{Definition eq_rect_r :}
\]

\[
\text{Coq} < \quad \text{forall (A:Type) (x:A) (P:A -> Type), P x -> forall y:A, y = x -> P y.}
\]

\[
\text{Coq} < \text{Hint Immediate sym_eq sym_not_eq : core.}
\]

The theorem \( f\_equal \) is extended to functions with two to five arguments. The theorem are names \( f\_equal2, f\_equal3, f\_equal4 \) and \( f\_equal5 \). For instance \( f\_equal3 \) is defined the following way.

\[
\text{Coq} < \text{Theorem f_equal3 :}
\]

\[
\text{Coq} < \quad \text{forall (A1 A2 A3 B:Type) (f:A1 -> A2 -> A3 -> B) (x1 y1:A1) (x2 y2:A2)}
\]

\[
\text{Coq} < \quad \text{(x3 y3:A3), x1 = y1 -> x2 = y2 -> x3 = y3 -> f x1 x2 x3 = f y1 y2 y3.}
\]

3.1.3 Datatypes

In the basic library, we find the definition\(^2\) of the basic data-types of programming, again defined as inductive constructions over the sort \( \text{Set} \). Some of them come with a special syntax shown on Figure 3.3.

\(^2\)They are in Datatypes.v
### 3.1 The basic library

<table>
<thead>
<tr>
<th>specif ::=</th>
<th>specif + specif</th>
<th>(sum)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>specif * specif</td>
<td>(prod)</td>
</tr>
<tr>
<td></td>
<td>{ specif } + { specif }</td>
<td>(sumbool)</td>
</tr>
<tr>
<td></td>
<td>(ident : specif</td>
<td>form)</td>
</tr>
<tr>
<td></td>
<td>(ident : specif</td>
<td>form &amp; form)</td>
</tr>
<tr>
<td></td>
<td>(ident : specif &amp; specif)</td>
<td>(sigT)</td>
</tr>
<tr>
<td></td>
<td>(ident : specif &amp; specif &amp; specif)</td>
<td>(sigT2)</td>
</tr>
</tbody>
</table>

| term ::= | (term, term) | (pair) |

---

#### Programming

Coq < Inductive unit : Set := tt.

Coq < Inductive bool : Set := true | false.


Coq < Inductive option (A:Set) : Set := Some (_:A) | None.

Coq < Inductive identity (A:Type) (a:A) : A -> Type :=

Coq < refl_identity : identity A a a.

Note that zero is the letter O, and not the numeral 0.

The predicate identity is logically equivalent to equality but it lives in sort Type. It is mainly maintained for compatibility.

We then define the disjoint sum of A+B of two sets A and B, and their product A*B.

Coq < Inductive sum (A B:Set) : Set := inl (_:A) | inr (_:B).

Coq < Inductive prod (A B:Set) : Set := pair (_:A) (_:B).

Coq < Section projections.

Coq < Variables A B : Set.

Coq < Definition fst (H: prod A B) := match H with

Coq < | pair x y => x

Coq < end.

Coq < Definition snd (H: prod A B) := match H with

Coq < | pair x y => y

Coq < end.

Coq < End projections.

Some operations on bool are also provided: andb (with infix notation \&\&), orb (with infix notation ||), xorb, implb and negb.

### 3.1.4 Specification

The following notions\(^3\) allow to build new data-types and specifications. They are available with the syntax shown on Figure 3.3.

\(^3\)They are defined in module Specif.v
For instance, given $A : Type$ and $P : A \to Prop$, the construct $\{x : A \mid P \, x\}$ (in abstract syntax $(\text{sig} \, A \, \text{P})$) is a Type. We may build elements of this set as $(\text{exist} \, x \, p)$ whenever we have a witness $x : A$ with its justification $p : P \, x$.

From such a $(\text{exist} \, x \, p)$ we may in turn extract its witness $x : A$ (using an elimination construct such as $\text{match}$) but not its justification, which stays hidden, like in an abstract data-type. In technical terms, one says that $\text{sig}$ is a “weak (dependent) sum”. A variant $\text{sig2}$ with two predicates is also provided.

Coq < Inductive sig2 (A:Set) (P Q:A -> Prop) : Set :=
Coq < exist2 (x:A) (_:P x) (_:Q x).

A “strong (dependent) sum” $\{x : A \& P \, x\}$ may be also defined, when the predicate $P$ is now defined as a constructor of types in $Type$.

Coq < Inductive sigT (A:Type) (P:A -> Type) : Type := existT (x:A) (_:P x).
Coq < Section Projections.
Coq < Variable A : Type.
Coq < Variable P : A -> Type.
Coq < Definition projT1 (H:sigT A P) := let (x, h) := H in x.
Coq < Definition projT2 (H:sigT A P) :=
Coq < match H return P (projT1 H) with
Coq < existT x h => h
Coq < end.
Coq < End Projections.
Coq < Inductive sigT2 (A: Type) (P Q:A -> Type) : Type :=
Coq < existT2 (x:A) (_:P x) (_:Q x).

A related non-dependent construct is the constructive sum $\{A\} + \{B\}$ of two propositions $A$ and $B$.

Coq < Inductive sumbool (A B:Prop) : Set := left (_:A) | right (_:B).

This $\text{sumbool}$ construct may be used as a kind of indexed boolean data-type. An intermediate between $\text{sumbool}$ and $\text{sum}$ is the mixed $\text{sumor}$ which combines $A:Set$ and $B:Prop$ in the $Set A+\{B\}$.

Coq < Inductive sumor (A:Set) (B:Prop) : Set := inleft (_:A) | inright (_:B).

We may define variants of the axiom of choice, like in Martin-Löf’s Intuitionistic Type Theory.

Coq < Lemma Choice :
Coq < forall (S S’:Set) (R:S -> S’ -> Prop),
Coq < (forall x:S, (y : S’ | R x y)) ->
Coq < (forall x:S, (y : S’ & R x y)).
Coq < Lemma Choice2 :
Coq < forall (S S’:Set) (R:S -> S’ -> Set),
Coq < (forall x:S, (y : S’ & R x y)) ->
3.1 The basic library

3.1.5 Basic Arithmetics

The basic library includes a few elementary properties of natural numbers, together with the definitions of predecessor, addition and multiplication. It also provides a scope nat_scope gathering standard notations for common operations (+, *) and a decimal notation for numbers. That is he can write 3 for (S (S (S O))). This also works on the left hand side of a match expression (see for example section 10.1). This scope is opened by default.

The following example is not part of the standard library, but it shows the usage of the notations:

Coq < Fixpoint even (n:nat) : bool :=  
Coq < match n with  
Coq < | 0 => true  
Coq < | 1 => false  
Coq < | S (S n) => even n  
Coq < end.

Coq < Theorem eq_S : forall x y:nat, x = y -> S x = S y.
Coq < Definition pred (n:nat) : nat :=  
Coq < match n with  
Coq < | 0 => 0  
Coq < | S u => u  
Coq < end.

Coq < Theorem pred_Sn : forall m:nat, m = pred (S m).
Coq < Theorem eq_add_S : forall n m:nat, S n = S m -> n = m.
Coq < Hint Immediate eq_add_S : core.
Coq < Theorem not_eq_S : forall n m:nat, n <> m -> S n <> S m.

4This is in module Peano.v
The Coq library

```
Coq < Definition IsSucc (n:nat) : Prop :=
Coq <  match n with
Coq <  | 0 => False
Coq <  | S p => True
Coq <  end. 
Coq < Theorem O_S : forall n:nat, 0 <> S n. 
Coq < Theorem n_Sn : forall n:nat, n <> S n. 
Coq < Fixpoint plus (n m:nat) {struct n} : nat :=
Coq <  match n with
Coq <  | 0 => m
Coq <  | S p => S (p + m)
Coq <  end.
Coq < where "n + m" := (plus n m) : nat_scope. 
Coq < Lemma plus_n_O : forall n:nat, n = n + 0. 
Coq < Lemma plus_n_Sm : forall n m:nat, S (n + m) = n + S m. 
Coq < Fixpoint mult (n m:nat) {struct n} : nat :=
Coq <  match n with
Coq <  | 0 => 0
Coq <  | S p => m + p * m
Coq <  end.
Coq < where "n * m" := (mult n m) : nat_scope. 
Coq < Lemma mult_n_O : forall n:nat, 0 = n * 0. 
Coq < Lemma mult_n_Sm : forall n m:nat, n * m + n = n * (S m). 
```

Finally, it gives the definition of the usual orderings \texttt{le}, \texttt{lt}, \texttt{ge}, and \texttt{gt}.

```
Coq < Inductive le (n:nat) : nat -> Prop :=
Coq <  le_n : le n n
Coq <  le_S : forall m:nat, n <= m -> n <= (S m).
Coq < where "n <= m" := (le n m) : nat_scope. 
Coq < Definition lt (n m:nat) := S n <= m. 
Coq < Definition ge (n m:nat) := m <= n. 
Coq < Definition gt (n m:nat) := m < n. 
```

Properties of these relations are not initially known, but may be required by the user from modules \texttt{Le} and \texttt{Lt}. Finally, \texttt{Peano} gives some lemmas allowing pattern-matching, and a double induction principle.

```
Coq < Theorem nat_case :
Coq <  forall (n:nat) (P:nat -> Prop), P 0 -> (forall m:nat, P (S m)) -> P n. 
Coq < Theorem nat_double_ind :
Coq <  forall R:nat -> nat -> Prop,
Coq <  (forall n:nat, R 0 n) ->
Coq <  (forall n:nat, R (S n) 0) ->
Coq <  (forall n m:nat, R n m -> R (S n) (S m)) -> forall n m:nat, R n m. 
```

Coq Reference Manual, V8.2p12, June 29, 2010
3.1 The basic library

3.1.6 Well-founded recursion

The basic library contains the basics of well-founded recursion and well-founded induction\(^5\).

Coq < Section Well_founded.
Coq < Variable A : Type.
Coq < Inductive Acc (x:A) : Prop :=
Coq < Acc_intro : (forall y:A, R y x -> Acc y) -> Acc x.
Coq < Lemma Acc_inv : Acc x -> forall y:A, R y x -> Acc y.
Coq < Definition well_founded := forall a:A, Acc a.
Coq < Hypothesis Rwf : well_founded.
Coq < Theorem well_founded_induction :
Coq < forall P:A -> Set,
Coq < (forall x:A, (forall y:A, R y x -> P y) -> P x) -> forall a:A, P a.
Coq < Theorem well_founded_ind :
Coq < forall P:A -> Prop,
Coq < (forall x:A, (forall y:A, R y x -> P y) -> P x) -> forall a:A, P a.

The automatically generated scheme \texttt{Acc_rect} can be used to define functions by fixpoints using well-founded relations to justify termination. Assuming extensionality of the functional used for the recursive call, the fixpoint equation can be proved.

Coq < Section FixPoint.
Coq < Variable P : A -> Type.
Coq < Variable F : forall x:A, (forall y:A, R y x -> P y) -> P x.
Coq < Fixpoint Fix_F (x:A) (r:Acc x) {struct r} : P x :=
Coq < F x (fun (y:A) (p:R y x) => Fix_F y (Acc_inv x r y p)).
Coq < Definition Fix (x:A) := Fix_F x (Rwf x).
Coq < Hypothesis F_ext :
Coq < forall (x:A) (f g:forall y:A, R y x -> P y),
Coq < (forall (y:A) (p:R y x), f y p = g y p) -> F x f = F x g.
Coq < Lemma Fix_F_eq :
Coq < forall (x:A) (r:Acc x),
Coq < F x (fun (y:A) (p:R y x) => Fix_F y (Acc_inv x r y p)) = Fix_F x r.
Coq < Lemma Fix_F_inv : forall (x:A) (r s:Acc x), Fix_F x r = Fix_F x s.
Coq < Lemma fix_eq : forall x:A, Fix x = F x (fun (y:A) (p:R y x) => Fix y).
Coq < End FixPoint.
Coq < End Well_founded.

\(^5\)This is defined in module \texttt{Wf.v}
3.1.7 Accessing the Type level

The basic library includes the definitions\(^6\) of the counterparts of some data-types and logical quantifiers at the Type level: negation, pair, and properties of identity.

Coq < Definition notT (A:Type) := A -> False.
Coq < Inductive prodT (A B:Type) : Type := pairT (_:A) (_:B).

At the end, it defines data-types at the Type level.

3.1.8 Tactics

A few tactics defined at the user level are provided in the initial state\(^7\).

3.2 The standard library

3.2.1 Survey

The rest of the standard library is structured into the following subdirectories:

- **Logic** Classical logic and dependent equality
- **Arith** Basic Peano arithmetic
- **NArith** Basic positive integer arithmetic
- **ZArith** Basic relative integer arithmetic
- **Numbers** Various approaches to natural, integer and cyclic numbers (currently axiomatically and on top of \(2^{31}\) binary words)
- **Bool** Booleans (basic functions and results)
- **Lists** Monomorphic and polymorphic lists (basic functions and results), Streams (infinite sequences defined with co-inductive types)
- **Sets** Sets (classical, constructive, finite, infinite, power set, etc.)
- **FSets** Specification and implementations of finite sets and finite maps (by lists and by AVL trees)
- **Reals** Axiomatization of real numbers (classical, basic functions, integer part, fractional part, limit, derivative, Cauchy series, power series and results, ...)
- **Relations** Relations (definitions and basic results)
- **Sorting** Sorted list (basic definitions and heapsort correctness)
- **Strings** 8-bits characters and strings
- **Wellfounded** Well-founded relations (basic results)

These directories belong to the initial load path of the system, and the modules they provide are compiled at installation time. So they are directly accessible with the command Require (see Chapter 6).

The different modules of the CoQ standard library are described in the additional document Library.dvi. They are also accessible on the WWW through the CoQ homepage\(^8\).

---

\(^6\)This is in module Logic_Type.v  
\(^7\)This is in module Tactics.v  
\(^8\)http://coq.inria.fr
3.2 The standard library

Notation | Interpretation | Precedence | Associativity
---|---|---|---
_ < _ | Zlt | | |
x <= y | Zle | | |
_< > _ | Zgt | | |
x >= y | Zge | | |
x < y < z | x < y \(\land\) y < z | | |
x < y <= z | x < y \(\land\) y <= z | | |
x <= y < z | x <= y \(\land\) y < z | | |
x <= y <= z | x <= y \(\land\) y <= z | | |
_ ?= _ | Zcompare | 70 | no
_ + _ | Zplus | | |
_ - _ | Zminus | | |
_ * _ | Zmult | | |
_ / _ | Zdiv | | |
_ mod _ | Zmod | 40 | no
_ ^ _ | Zpower | | |

Figure 3.4: Definition of the scope for integer arithmetics (Z_scope)

3.2.2 Notations for integer arithmetics

On Figure 3.2.2 is described the syntax of expressions for integer arithmetics. It is provided by requiring and opening the module ZArith and opening scope Z_scope.

Figure 3.2.2 shows the notations provided by Z_scope. It specifies how notations are interpreted and, when not already reserved, the precedence and associativity.

Coq < Require Import ZArith.
Coq < Check (2 + 3)%Z.
(2 + 3)%Z : Z
Coq < Open Scope Z_scope.
Coq < Check 2 + 3.
2 + 3 : Z

3.2.3 Peano’s arithmetic (nat)

While in the initial state, many operations and predicates of Peano’s arithmetic are defined, further operations and results belong to other modules. For instance, the decidability of the basic predicates are defined here. This is provided by requiring the module Arith.

Figure 3.2.3 describes notation available in scope nat_scope.
3.2.4 Real numbers library

Notations for real numbers

This is provided by requiring and opening the module `Reals` and opening scope `R_scope`. This set of notations is very similar to the notation for integer arithmetics. The inverse function was added.

```coq
Coq < Require Import Reals.
Coq < Check (2 + 3)%R.
(2 + 3)%R : R
Coq < Open Scope R_scope.
Coq < Check 2 + 3.
```
Some tactics

In addition to the ring, field and fourier tactics (see Chapter 8) there are:

- **discrR**
  
  Proves that a real integer constant \( c_1 \) is different from another real integer constant \( c_2 \).

  ```coq
  Coq < Require Import DiscrR.
  Coq < Goal 5 <> 0.
  Coq < discrR.
  Proof completed.
  ```

- **split_Rabs** allows to unfold \( Rabs \) constant and splits corresponding conjunctions.

  ```coq
  Coq < Require Import SplitAbsolu.
  Coq < Goal forall x:R, x <= Rabs x.
  Coq < intro; split_Rabs.
  2 subgoals
  x : R
  r : x < 0
  =============================
  x <= - x
  subgoal 2 is:
  x <= x
  ```

- **split_Rmult** allows to split a condition that a product is non null into subgoals corresponding to the condition on each operand of the product.

  ```coq
  Coq < Require Import SplitRmult.
  Coq < Goal forall x y z:R, x * y * z <> 0.
  Coq < intros; split_Rmult.
  3 subgoals
  x : R
  y : R
  z : R
  =============================
  x <> 0
  subgoal 2 is:
  y <> 0
  subgoal 3 is:
  z <> 0
  ```

All this tactics has been written with the tactic language Ltac described in Chapter 9. More details are available in document [http://coq.inria.fr/~desmettr/Reals.ps](http://coq.inria.fr/~desmettr/Reals.ps).
3.2.5 List library

Some elementary operations on polymorphic lists are defined here. They can be accessed by requiring module List.

It defines the following notions:

<table>
<thead>
<tr>
<th>Notation</th>
<th>Interpretation</th>
<th>Precedence</th>
<th>Associativity</th>
</tr>
</thead>
<tbody>
<tr>
<td>_ ++ _</td>
<td>app</td>
<td>60</td>
<td>right</td>
</tr>
<tr>
<td>_ :: _</td>
<td>cons</td>
<td>60</td>
<td>right</td>
</tr>
</tbody>
</table>

Figure 3.7: Definition of the scope for lists (list_scope)

3.3 Users’ contributions

Numerous users’ contributions have been collected and are available at URL coq.inria.fr/contribs/. On this web page, you have a list of all contributions with informations (author, institution, quick description, etc.) and the possibility to download them one by one. There is a small search engine to look for keywords in all contributions. You will also find informations on how to submit a new contribution.

The users’ contributions may also be obtained by anonymous FTP from site ftp.inria.fr, in directory INRIA/coq/ and searchable on-line at http://coq.inria.fr/contribs-eng.html
Chapter 4

Calculus of Inductive Constructions

The underlying formal language of CoQ is a Calculus of Constructions with Inductive Definitions. It is presented in this chapter. For CoQ version V7, this Calculus was known as the Calculus of (Co)Inductive Constructions (CIC in short). The underlying calculus of CoQ version V8.0 and up is a weaker calculus where the sort Set satisfies predicative rules. We call this calculus the Predicative Calculus of (Co)Inductive Constructions (pCIC in short). In Section 4.7 we give the extra-rules for CIC. A compiling option of CoQ allows to type-check theories in this extended system.

In pCIC all objects have a type. There are types for functions (or programs), there are atomic types (especially datatypes)... but also types for proofs and types for the types themselves. Especially, any object handled in the formalism must belong to a type. For instance, the statement “for all x, P” is not allowed in type theory; you must say instead: “for all x belonging to T, P”. The expression “x belonging to T” is written “x:T”. One also says: “x has type T”. The terms of pCIC are detailed in Section 4.1.

In pCIC there is an internal reduction mechanism. In particular, it allows to decide if two programs are intentionally equal (one says convertible). Convertibility is presented in section 4.3.

The remaining sections are concerned with the type-checking of terms. The beginner can skip them.

The reader seeking a background on the Calculus of Inductive Constructions may read several papers. Giménez and Castéran [68] provide an introduction to inductive and coinductive definitions in Coq. In their book [14], Bertot and Castéran give a precise description of the pCIC based on numerous practical examples. Barras [9], Werner [135] and Paulin-Mohring [117] are the most recent theses dealing with Inductive Definitions. Coquand-Huet [29, 30, 31] introduces the Calculus of Constructions. Coquand-Paulin [32] extended this calculus to inductive definitions. The pCIC is a formulation of type theory including the possibility of inductive constructions, Barendregt [6] studies the modern form of type theory.

4.1 The terms

In most type theories, one usually makes a syntactic distinction between types and terms. This is not the case for pCIC which defines both types and terms in the same syntactical structure. This is because the type-theory itself forces terms and types to be defined in a mutual recursive way and also because similar constructions can be applied to both terms and types and consequently can share the same syntactic structure.

Consider for instance the → constructor and assume nat is the type of natural numbers. Then → is used both to denote nat → nat which is the type of functions from nat to nat, and to denote nat → Prop which is the type of unary predicates over the natural numbers. Consider abstraction which
builds functions. It serves to build “ordinary” functions as \( \text{fun } x : \text{nat} \Rightarrow (\text{mult } x x) \) (assuming \( \text{mult} \) is already defined) but may build also predicates over the natural numbers. For instance \( \text{fun } x : \text{nat} \Rightarrow (x = x) \) will represent a predicate \( P \), informally written in mathematics \( P(x) \equiv x = x \). If \( P \) has type \( \text{nat} \Rightarrow \text{Prop} \), \( (P x) \) is a proposition, furthermore for all \( x : \text{nat} \), \( (P x) \) will represent the type of functions which associate to each natural number \( n \) an object of type \( (P n) \) and consequently represent proofs of the formula “\( \forall x. P(x) \)”.

4.1.1 Sorts

Types are seen as terms of the language and therefore should belong to another type. The type of a type is always a constant of the language called a sort.

The two basic sorts in the language of \( \text{pCic} \) are \( \text{Set} \) and \( \text{Prop} \).

The sort \( \text{Prop} \) intends to be the type of logical propositions. If \( M \) is a logical proposition then it denotes a class, namely the class of terms representing proofs of \( M \). An object \( m \) belonging to \( M \) witnesses the fact that \( M \) is true. An object of type \( \text{Prop} \) is called a proposition.

The sort \( \text{Set} \) intends to be the type of specifications. This includes programs and the usual sets such as booleans, naturals, lists etc.

These sorts themselves can be manipulated as ordinary terms. Consequently sorts also should be given a type. Because assuming simply that \( \text{Set} \) has type \( \text{Set} \) leads to an inconsistent theory, we have infinitely many sorts in the language of \( \text{pCic} \). These are, in addition to \( \text{Set} \) and \( \text{Prop} \) a hierarchy of universes \( \text{Type}(i) \) for any integer \( i \). We call \( S \) the set of sorts which is defined by:

\[
S \equiv \{ \text{Prop}, \text{Set}, \text{Type}(i) \mid i \in \mathbb{N} \}
\]

The sorts enjoy the following properties: \( \text{Prop} : \text{Type}(0) \), \( \text{Set} : \text{Type}(0) \) and \( \text{Type}(i) : \text{Type}(i + 1) \).

The user will never mention explicitly the index \( i \) when referring to the universe \( \text{Type}(i) \). One only writes \( \text{Type} \). The system itself generates for each instance of \( \text{Type} \) a new index for the universe and checks that the constraints between these indexes can be solved. From the user point of view we consequently have \( \text{Type} : \text{Type} \).

We shall make precise in the typing rules the constraints between the indexes.

**Implementation issues** In practice, the \( \text{Type} \) hierarchy is implemented using algebraic universes. An algebraic universe \( u \) is either a variable (a qualified identifier with a number) or a successor of an algebraic universe (an expression \( u + 1 \)), or an upper bound of algebraic universes (an expression \( \text{max}(u_1, \ldots, u_n) \)), or the base universe (the expression 0) which corresponds, in the arity of sort-polymorphic inductive types, to the predicative sort \( \text{Set} \). A graph of constraints between the universe variables is maintained globally. To ensure the existence of a mapping of the universes to the positive integers, the graph of constraints must remain acyclic. Typing expressions that violate the acyclicity of the graph of constraints results in a **Universe inconsistency** error (see also Section 2.10).

4.1.2 Constants

Besides the sorts, the language also contains constants denoting objects in the environment. These constants may denote previously defined objects but also objects related to inductive definitions (either the type itself or one of its constructors or destructors).

**Remark.** In other presentations of \( \text{pCic} \), the inductive objects are not seen as external declarations but as first-class terms. Usually the definitions are also completely ignored. This is a nice theoretical point
of view but not so practical. An inductive definition is specified by a possibly huge set of declarations, clearly we want to share this specification among the various inductive objects and not to duplicate it. So the specification should exist somewhere and the various objects should refer to it. We choose one more level of indirection where the objects are just represented as constants and the environment gives the information on the kind of object the constant refers to.

Our inductive objects will be manipulated as constants declared in the environment. This roughly corresponds to the way they are actually implemented in the COQ system. It is simple to map this presentation in a theory where inductive objects are represented by terms.

### 4.1.3 Terms

Terms are built from variables, global names, constructors, abstraction, application, local declarations bindings (“let-in” expressions) and product.

From a syntactic point of view, types cannot be distinguished from terms, except that they cannot start by an abstraction, and that if a term is a sort or a product, it should be a type.

More precisely the language of the Calculus of Inductive Constructions is built from the following rules:

1. the sorts Set, Prop, Type are terms.
2. names for global constants of the environment are terms.
3. variables are terms.
4. if \( x \) is a variable and \( T, U \) are terms then \( \forall x : T, U \) (forall \( x : T, U \) in COQ concrete syntax) is a term. If \( x \) occurs in \( U \), \( \forall x : T, U \) reads as “for all \( x \) of type \( T, U \)”. As \( U \) depends on \( x \), one says that \( \forall x : T, U \) is a dependent product. If \( x \) doesn’t occurs in \( U \) then \( \forall x : T, U \) reads as “if \( T \) then \( U \)”. A non dependent product can be written: \( T \to U \).
5. if \( x \) is a variable and \( T, U \) are terms then \( \lambda x : T, U \) (fun \( x : T \Rightarrow U \) in COQ concrete syntax) is a term. This is a notation for the \( \lambda \)-abstraction of \( \lambda \)-calculus [8]. The term \( \lambda x : T, U \) is a function which maps elements of \( T \) to \( U \).
6. if \( T \) and \( U \) are terms then \( (T U) \) is a term (\( T U \) in COQ concrete syntax). The term \( (T U) \) reads as “\( T \) applied to \( U \)”.
7. if \( x \) is a variable, and \( T, U \) are terms then let \( x := T \) in \( U \) is a term which denotes the term \( U \) where the variable \( x \) is locally bound to \( T \). This stands for the common “let-in” construction of functional programs such as ML or Scheme.

**Notations.** Application associates to the left such that \( (t_1 \ldots t_n) \) represents \( (\ldots (t_1) \ldots t_n) \). The products and arrows associate to the right such that \( \forall x : A, B \to C \to D \) represents \( \forall x : A, (B \to (C \to D)) \). One uses sometimes \( \forall x \ y : A, B \) or \( \lambda x \ y : A, B \) to denote the abstraction or product of several variables of the same type. The equivalent formulation is \( \forall x : A, \forall y : A, B \) or \( \lambda x : A, \lambda y : A, B \)

**Free variables.** The notion of free variables is defined as usual. In the expressions \( \lambda x : T, U \) and \( \forall x : T, U \) the occurrences of \( x \) in \( U \) are bound. They are represented by de Bruijn indexes in the internal structure of terms.
Substitution. The notion of substituting a term \( t \) to free occurrences of a variable \( x \) in a term \( u \) is defined as usual. The resulting term is written \( u\{x/t\} \).

4.2 Typed terms

As objects of type theory, terms are subjected to type discipline. The well typing of a term depends on an environment which consists in a global environment (see below) and a local context.

**Local context.** A local context (or shortly context) is an ordered list of declarations of variables. The declaration of some variable \( x \) is either an assumption, written \( x : T \) (\( T \) is a type) or a definition, written \( x := t : T \). We use brackets to write contexts. A typical example is \([x : T; y := u : U; z : V]\). Notice that the variables declared in a context must be distinct. If \( \Gamma \) declares some \( x \), we write \( x \in \Gamma \). By writing \( (x : T) \in \Gamma \) we mean that either \( x : T \) is an assumption in \( \Gamma \) or that there exists some \( t \) such that \( x := t : T \) is a definition in \( \Gamma \). If \( \Gamma \) defines some \( x := t : T \), we also write \( (x := t : T) \in \Gamma \).

Contexts must be themselves well formed. For the rest of the chapter, the notation \( \Gamma :: (y := t : T) \) (resp. \( \Gamma :: (y := t : T) ) \) denotes the context \( \Gamma \) enriched with the declaration \( y : T \) (resp. \( y := t : T \)). The notation \([\] \) denotes the empty context.

We define the inclusion of two contexts \( \Gamma \) and \( \Delta \) (written as \( \Gamma \subset \Delta \)) as the property, for all variable \( x \), type \( T \) and term \( t \), if \( (x : T) \in \Gamma \) then \( (x : T) \in \Delta \) and if \( (x := t : T) \in \Gamma \) then \( (x := t : T) \in \Delta \).

A variable \( x \) is said to be free in \( \Gamma \) if \( \Gamma \) contains a declaration \( y : T \) such that \( x \) is free in \( T \).

**Environment.** Because we are manipulating global declarations (constants and global assumptions), we also need to consider a global environment \( E \).

An environment is an ordered list of declarations of global names. Declarations are either assumptions or “standard” definitions, that is abbreviations for well-formed terms but also definitions of inductive objects. In the latter case, an object in the environment will define one or more constants (that is types and constructors, see Section 4.5).

An assumption will be represented in the environment as \( \text{Assum}(\Gamma) \{ c : T \} \) which means that \( c \) is assumed of some type \( T \) well-defined in some context \( \Gamma \). An (ordinary) definition will be represented in the environment as \( \text{Def}(\Gamma) \{ c := t : T \} \) which means that \( c \) is a constant which is valid in some context \( \Gamma \) whose value is \( t \) and type is \( T \).

The rules for inductive definitions (see section 4.5) have to be considered as assumption rules to which the following definitions apply: if the name \( c \) is declared in \( E \), we write \( c \in E \) and if \( c : T \) or \( c := t : T \) is declared in \( E \), we write \( (c : T) \in E \).

**Typing rules.** In the following, we assume \( E \) is a valid environment wrt to inductive definitions. We define simultaneously two judgments. The first one \( E[\Gamma] \vdash t : T \) means the term \( t \) is well-typed and has type \( T \) in the environment \( E \) and context \( \Gamma \). The second judgment \( \forall \varphi(E)[\Gamma] \) means that the environment \( E \) is well-formed and the context \( \Gamma \) is a valid context in this environment. It also means a third property which makes sure that any constant in \( E \) was defined in an environment which is included in \( \Gamma \).

A term \( t \) is well typed in an environment \( E \) iff there exists a context \( \Gamma \) and a term \( T \) such that the judgment \( E[\Gamma] \vdash t : T \) can be derived from the following rules.

---

1This requirement could be relaxed if we instead introduced an explicit mechanism for instantiating constants. At the external level, the Coq engine works accordingly to this view that all the definitions in the environment were built in a sub-context of the current context.
4.2 Typed terms

W-E
\[ \mathcal{WF}(\text{[]})[\text{[]}] \]

W-S
\[ \begin{array}{c}
E[\Gamma] \vdash T : s \quad s \in S \quad x \not\in \Gamma \\
\mathcal{WF}(E)[\Gamma :: (x : T)] \\
\hline
E[\Gamma] \vdash t : T \quad x \not\in \Gamma \\
\mathcal{WF}(E)[\Gamma :: (x := t : T)]
\end{array} \]

Def
\[ E[\Gamma] \vdash t : T \quad c \not\in E \cup \Gamma \\
\mathcal{WF}(E; \text{Def}(\Gamma)(c := t : T))[\Gamma] \]

Assum
\[ E[\Gamma] \vdash T : s \quad s \in S \quad c \not\in E \cup \Gamma \\
\mathcal{WF}(E; \text{Assum}(\Gamma)(c : T))[\Gamma] \]

Ax
\[ \begin{array}{c}
\mathcal{WF}(E)[\Gamma] \\
E[\Gamma] \vdash \text{Prop} : \text{Type}(p) \quad E[\Gamma] \vdash \text{Set} : \text{Type}(q) \\
\mathcal{WF}(E)[\Gamma] \\
i < j \\
\mathcal{WF}(E)[\Gamma] \\
E[\Gamma] \vdash \text{Type}(i) : \text{Type}(j) \\
\end{array} \]

Var
\[ \mathcal{WF}(E)[\Gamma] \\
(x : T) \in \Gamma \text{ or } (x := t : T) \in \Gamma \text{ for some } t \\
E[\Gamma] \vdash x : T \]

Const
\[ \mathcal{WF}(E)[\Gamma] \\
c : T \in E \text{ or } (c := t : T) \in E \text{ for some } t \\
E[\Gamma] \vdash c : T \]

Prod
\[ \begin{array}{c}
E[\Gamma] \vdash T : s \quad s \in S \\
E[\Gamma] \vdash \forall x : T, U : \text{Prop} \\
E[\Gamma] \vdash T : s \quad s \in \{\text{Prop, Set}\} \\
E[\Gamma] \vdash \forall x : T, U : \text{Set} \\
E[\Gamma] \vdash T : \text{Type}(i) \quad i \leq k \\
E[\Gamma] \vdash \forall x : T, U : \text{Type}(j) \quad j \leq k \\
E[\Gamma] \vdash \forall x : T, U : \text{Type}(k)
\end{array} \]

Lam
\[ \begin{array}{c}
E[\Gamma] \vdash \forall x : T, U : s \\
E[\Gamma] \vdash (x : T) \vdash t : U \\
E[\Gamma] \vdash \lambda x : T, t : \forall x : T, U
\end{array} \]

App
\[ \begin{array}{c}
E[\Gamma] \vdash t : \forall x : U, T \\
E[\Gamma] \vdash u : U \\
E[\Gamma] \vdash (t \ u) : T\{x/u\}
\end{array} \]

Let
\[ \begin{array}{c}
E[\Gamma] \vdash T \\
E[\Gamma] \vdash (x := t : T) \vdash u : U \\
E[\Gamma] \vdash \text{let } x := t \text{ in } u : U\{x/t\}
\end{array} \]

Remark: We may have \text{let } x := t \text{ in } u \text{ well-typed without having } ((\lambda x : T, u) t) \text{ well-typed (where } T \text{ is a type of } t). \text{ This is because the value } t \text{ associated to } x \text{ may be used in a conversion rule (see Section 4.3).}
4.3 Conversion rules

**β-reduction.** We want to be able to identify some terms as we can identify the application of a function to a given argument with its result. For instance the identity function over a given type $T$ can be written $\lambda x : T, x$. In any environment $E$ and context $\Gamma$, we want to identify any object $a$ (of type $T$) with the application $((\lambda x : T, t) \ a)$. We define for this a reduction (or a conversion) rule we call $\beta$:

$$E[\Gamma] \vdash ((\lambda x : T, t) \ u) \triangleright_\beta t \{x/u\}$$

We say that $t \{x/u\}$ is the $\beta$-contraction of $((\lambda x : T, t) \ u)$ and, conversely, that $((\lambda x : T, t) \ u)$ is the $\beta$-expansion of $t \{x/u\}$.

According to $\beta$-reduction, terms of the Calculus of Inductive Constructions enjoy some fundamental properties such as confluence, strong normalization, subject reduction. These results are theoretically of great importance but we will not detail them here and refer the interested reader to [23].

**ι-reduction.** A specific conversion rule is associated to the inductive objects in the environment. We shall give later on (see Section 4.5.4) the precise rules but it just says that a destructor applied to an object built from a constructor behaves as expected. This reduction is called $\iota$-reduction and is more precisely studied in [116, 135].

**δ-reduction.** We may have defined variables in contexts or constants in the global environment. It is legal to identify such a reference with its value, that is to expand (or unfold) it into its value. This reduction is called $\delta$-reduction and shows as follows.

$$E[\Gamma] \vdash x \triangleright_\delta t \quad \text{if} \quad (x := t : T) \in \Gamma \quad E[\Gamma] \vdash c \triangleright_\delta t \quad \text{if} \quad (c := t : T) \in E$$

**ζ-reduction.** Coq allows also to remove local definitions occurring in terms by replacing the defined variable by its value. The declaration being destroyed, this reduction differs from $\delta$-reduction. It is called $\zeta$-reduction and shows as follows.

$$E[\Gamma] \vdash \text{let } x := u \text{ in } t \triangleright_\zeta t \{x/u\}$$

**Convertibility.** Let us write $E[\Gamma] \vdash t \triangleright u$ for the contextual closure of the relation $t$ reduces to $u$ in the environment $E$ and context $\Gamma$ with one of the previous reduction $\beta, \iota, \delta$ or $\zeta$.

We say that two terms $t_1$ and $t_2$ are convertible (or equivalent) in the environment $E$ and context $\Gamma$ iff there exists a term $u$ such that $E[\Gamma] \vdash t_1 \triangleright \ldots \triangleright u$ and $E[\Gamma] \vdash t_2 \triangleright \ldots \triangleright u$. We then write $E[\Gamma] \vdash t_1 =_{\beta\iota\delta\zeta} t_2$.

The convertibility relation allows to introduce a new typing rule which says that two convertible well-formed types have the same inhabitants.

At the moment, we did not take into account one rule between universes which says that any term in a universe of index $i$ is also a term in the universe of index $i + 1$. This property is included into the conversion rule by extending the equivalence relation of convertibility into an order inductively defined by:

1. if $E[\Gamma] \vdash t =_{\beta\iota\delta\zeta} u$ then $E[\Gamma] \vdash t \leq_{\beta\iota\delta\zeta} u$,
2. if $i \leq j$ then $E[\Gamma] \vdash \text{Type}(i) \leq_{\beta\iota\delta\zeta} \text{Type}(j)$.
4.4 Derived rules for environments

3. for any $i$, $E[\Gamma] \vdash \text{Prop} \leq_{\beta\delta\iota} \text{Type}(i)$,

4. for any $i$, $E[\Gamma] \vdash \text{Set} \leq_{\beta\delta\iota} \text{Type}(i)$,

5. $E[\Gamma] \vdash \text{Prop} \leq_{\beta\delta\iota} \text{Set}$,

6. if $E[\Gamma] \vdash T =_{\beta\delta\iota} U$ and $E[\Gamma :: (x : T)] \vdash T' \leq_{\beta\delta\iota} U'$

then

The conversion rule is now exactly:

$$
\text{Conv} \\
E[\Gamma] \vdash U : s \quad E[\Gamma] \vdash t : T \quad E[\Gamma] \vdash T \leq_{\beta\delta\iota} U \\
\frac{}{E[\Gamma] \vdash t : U}
$$

**\eta-conversion.** An other important rule is the \eta-conversion. It is to identify terms over a dummy abstraction of a variable followed by an application of this variable. Let $T$ be a type, $t$ be a term in which the variable $x$ doesn’t occurs free. We have

$$E[\Gamma] \vdash \lambda x : T, (t x) \triangleright t$$

Indeed, as $x$ doesn’t occur free in $t$, for any $u$ one applies to $\lambda x : T, (t x)$, it $\beta$-reduces to $(t u)$. So $\lambda x : T, (t x)$ and $t$ can be identified.

**Remark:** The $\eta$-reduction is not taken into account in the convertibility rule of CoQ.

**Normal form.** A term which cannot be any more reduced is said to be in normal form. There are several ways (or strategies) to apply the reduction rule. Among them, we have to mention the head reduction which will play an important role (see Chapter 8). Any term can be written as $\lambda x_1 : T_1, \ldots, \lambda x_k : T_k, (t_0 \ t_1 \ldots t_n)$ where $t_0$ is not an application. We say then that $t_0$ is the head of $t$. If we assume that $t_0$ is $\lambda x : T, u_0$ then one step of $\beta$-head reduction of $t$ is:

$$\lambda x_1 : T_1, \ldots, \lambda x_k : T_k, (\lambda x : T, u_0 \ t_1 \ldots t_n) \triangleright \lambda (x_1 : T_1) \ldots (x_k : T_k), (u_0 \{x/t_1\} \ t_2 \ldots t_n)$$

Iterating the process of head reduction until the head of the reduced term is no more an abstraction leads to the $\beta$-head normal form of $t$:

$$t \triangleright \ldots \triangleright \lambda x_1 : T_1, \ldots, \lambda x_k : T_k, (v \ u_1 \ldots u_m)$$

where $v$ is not an abstraction (nor an application). Note that the head normal form must not be confused with the normal form since some $u_i$ can be reducible.

Similar notions of head-normal forms involving $\delta$, $\iota$ and $\zeta$ reductions or any combination of those can also be defined.

4.4 Derived rules for environments

From the original rules of the type system, one can derive new rules which change the context of definition of objects in the environment. Because these rules correspond to elementary operations in the CoQ engine used in the discharge mechanism at the end of a section, we state them explicitly.
Mechanism of substitution. One rule which can be proved valid, is to replace a term \( c \) by its value in the environment. As we defined the substitution of a term for a variable in a term, one can define the substitution of a term for a constant. One easily extends this substitution to contexts and environments.

Substitution Property:

\[
WF(E; Def(\Gamma)(c := t : T); F)[\Delta] \\
WF(E; F[c/t])[\Delta{c/t}]
\]

Abstraction. One can modify the context of definition of a constant \( c \) by abstracting a constant with respect to the last variable \( x \) of its defining context. For doing that, we need to check that the constants appearing in the body of the declaration do not depend on \( x \), we need also to modify the reference to the constant \( c \) in the environment and context by explicitly applying this constant to the variable \( x \). Because of the rules for building environments and terms we know the variable \( x \) is available at each stage where \( c \) is mentioned.

Abstracting property:

\[
WF(E; Def(\Gamma :: (x : U))(c := t : T); F)[\Delta] \\
WF(E; Def(\Gamma)(c := \lambda x : U, t : \forall x : U, T); F[c/(c x)]')[\Delta{c/(c x)}]
\]

Pruning the context. We said the judgment \( WF(E)[\Gamma] \) means that the defining contexts of constants in \( E \) are included in \( \Gamma \). If one abstracts or substitutes the constants with the above rules then it may happen that the context \( \Gamma' \) is now bigger than the one needed for defining the constants in \( E \). Because defining contexts are growing in \( E \), the minimum context needed for defining the constants in \( E \) is the same as the one for the last constant. One can consequently derive the following property.

Pruning property:

\[
WF(E; Def(\Delta)(c := t : T))[\Gamma] \\
WF(E; Def(\Delta)(c := t : T))[\Delta]
\]

4.5 Inductive Definitions

A (possibly mutual) inductive definition is specified by giving the names and the type of the inductive sets or families to be defined and the names and types of the constructors of the inductive predicates. An inductive declaration in the environment can consequently be represented with two contexts (one for inductive definitions, one for constructors).

Stating the rules for inductive definitions in their general form needs quite tedious definitions. We shall try to give a concrete understanding of the rules by precising them on running examples. We take as examples the type of natural numbers, the type of parameterized lists over a type \( A \), the relation which states that a list has some given length and the mutual inductive definition of trees and forests.

4.5.1 Representing an inductive definition

Inductive definitions without parameters

As for constants, inductive definitions can be defined in a non-empty context. We write \( \text{Ind}(\Gamma)(\Gamma_I := \Gamma_C) \) an inductive definition valid in a context \( \Gamma \), a context of definitions \( \Gamma_I \) and a context of constructors \( \Gamma_C \).
Examples. The inductive declaration for the type of natural numbers will be:

\[
\text{Ind}(\text{nat} : \text{Set} := \text{O} : \text{nat}, \text{S} : \text{nat} \to \text{nat})
\]

In a context with a variable \(A : \text{Set}\), the lists of elements in \(A\) are represented by:

\[
\text{Ind}(A : \text{Set}) (\text{List} : \text{Set} := \text{nil} : \text{List}, \text{cons} : A \to \text{List} \to \text{List})
\]

Assuming \(\Gamma_I\) is \([I_1 : A_1; \ldots; I_k : A_k]\), and \(\Gamma_C\) is \([c_1 : C_1; \ldots; c_n : C_n]\), the general typing rules are, for \(1 \leq j \leq k\) and \(1 \leq i \leq n\):

\[
\begin{align*}
\text{Ind}(\Gamma)(\Gamma_I := \Gamma_C) & \in E \\
(I_j : A_j) & \in E \\
\text{Ind}(\Gamma)(\Gamma_I := \Gamma_C) & \in E \\
(c_i : C_i) & \in E
\end{align*}
\]

Inductive definitions with parameters

We have to slightly complicate the representation above in order to handle the delicate problem of parameters. Let us explain that on the example of \(\text{List}\). With the above definition, the type \(\text{List}\) can only be used in an environment where we have a variable \(A : \text{Set}\). Generally one wants to consider lists of elements in different types. For constants this is easily done by abstracting the value over the parameter. In the case of inductive definitions we have to handle the abstraction over several objects.

One possible way to do that would be to define the type \(\text{List}\) inductively as being an inductive family of type \(\text{Set} \to \text{Set}\):

\[
\text{Ind}(\text{List} : \text{Set} \to \text{Set} := \text{nil} : \text{Set}, \text{cons} : (\forall A : \text{Set}, A \to \text{List} \to \text{List}))
\]

There are drawbacks to this point of view. The information which says that for any \(A\), \((\text{List} A)\) is an inductively defined \(\text{Set}\) has been lost. So we introduce two important definitions.

Inductive parameters, real arguments. An inductive definition \(\text{Ind}(\Gamma)(\Gamma_I := \Gamma_C)\) admits \(r\) inductive parameters if each type of constructors \((c : C)\) in \(\Gamma_C\) is such that

\[
C \equiv \forall p_1 : P_1, \ldots, \forall p_r : P_r, \forall a_1 : A_1, \ldots, \forall a_n : A_n, (I \ p_1 \ldots p_r \ t_1 \ldots t_q)
\]

with \(I\) one of the inductive definitions in \(\Gamma_I\). We say that \(q\) is the number of real arguments of the constructor \(c\).

Context of parameters. If an inductive definition \(\text{Ind}(\Gamma)(\Gamma_I := \Gamma_C)\) admits \(r\) inductive parameters, then there exists a context \(\Gamma_P\) of size \(r\), such that \(\Gamma_P = [p_1 : P_1; \ldots; p_r : P_r]\) and if \((t : A) \in \Gamma_I, \Gamma_C\) then \(A\) can be written as \(\forall p_1 : P_1, \ldots, \forall p_r : P_r, A'\). We call \(\Gamma_P\) the context of parameters of the inductive definition and use the notation \(\forall \Gamma_P, A'\) for the term \(A\).

Remark. If we have a term \(t\) in an instance of an inductive definition \(I\) which starts with a constructor \(c\), then the \(r\) first arguments of \(c\) (the parameters) can be deduced from the type \(T\) of \(t\): these are exactly the \(r\) first arguments of \(I\) in the head normal form of \(T\).
Examples. The List definition has 1 parameter:

\[
\text{Ind}()\left(\text{List} : \text{Set} \rightarrow \text{Set} := \text{nil} : (\forall A : \text{Set}, \text{List} A), \text{cons} : (\forall A : \text{Set}, A \rightarrow \text{List} A \rightarrow \text{List} A)\right)
\]

This is also the case for this more complex definition where there is a recursive argument on a different instance of List:

\[
\text{Ind}()\left(\text{List} : \text{Set} \rightarrow \text{Set} := \text{nil} : (\forall A : \text{Set}, \text{List} A), \text{cons} : (\forall A : \text{Set}, A \rightarrow \text{List} (A \rightarrow A) \rightarrow \text{List} A)\right)
\]

But the following definition has 0 parameters:

\[
\text{Ind}()\left(\text{List} : \text{Set} \rightarrow \text{Set} := \text{nil} : (\forall A : \text{Set}, \text{List} A), \text{cons} : (\forall A : \text{Set}, A \rightarrow \text{List} A \rightarrow \text{List} (A \ast A))\right)
\]

Concrete syntax. In the Coq system, the context of parameters is given explicitly after the name of the inductive definitions and is shared between the arities and the type of constructors. We keep track in the syntax of the number of parameters.

Formally the representation of an inductive declaration will be \(\text{Ind}(\Gamma)[p](\Gamma_I := \Gamma_C)\) for an inductive definition valid in a context \(\Gamma\) with \(p\) parameters, a context of definitions \(\Gamma_I\) and a context of constructors \(\Gamma_C\).

The definition \(\text{Ind}(\Gamma)[p](\Gamma_I := \Gamma_C)\) will be well-formed exactly when \(\text{Ind}(\Gamma)(\Gamma_I := \Gamma_C)\) is and when \(p\) is (less or equal than) the number of parameters in \(\text{Ind}(\Gamma)(\Gamma_I := \Gamma_C)\).

Examples The declaration for parameterized lists is:

\[
\text{Ind}()[1]\left(\text{List} : \text{Set} \rightarrow \text{Set} := \text{nil} : (\forall A : \text{Set}, \text{List} A), \text{cons} : (\forall A : \text{Set}, A \rightarrow \text{List} A \rightarrow \text{List} A)\right)
\]

The declaration for the length of lists is:

\[
\text{Ind}()[1]\left(\text{Length} : \forall A : \text{Set}, (\text{List} A) \rightarrow \text{nat} \rightarrow \text{Prop} := \text{Lnil} : \forall A : \text{Set}, \text{Length} A (\text{nil} A) \text{O}, \text{Lcons} : \forall A : \text{Set}, \forall a : A, \forall l : (\text{List} A), \forall n : \text{nat}, (\text{Length} A l n) \rightarrow (\text{Length} A (\text{cons} A a l) (\text{S} n))\right)
\]

The declaration for a mutual inductive definition of forests and trees is:

\[
\text{Ind}()\left(\text{tree} : \text{Set} \rightarrow \text{forest} : \text{Set} := \text{node} : \text{forest} \rightarrow \text{tree}, \text{emptyf} : \text{forest}, \text{consf} : \text{tree} \rightarrow \text{forest} \rightarrow \text{forest}\right)
\]

These representations are the ones obtained as the result of the CoQ declaration:

\[
\text{Coq} < \text{Inductive} \text{nat} : \text{Set} := \text{O} : \text{nat} \mid \text{S} : \text{nat} \rightarrow \text{nat}.
\text{Coq} < \text{Inductive} \text{list} (A:\text{Set}) : \text{Set} := \text{nil} : \text{list} A \mid \text{cons} : A \rightarrow \text{list} A \rightarrow \text{list} A.
\text{Coq} < \text{Inductive} \text{Length} (A:\text{Set}) : \text{list} A \rightarrow \text{nat} \rightarrow \text{Prop} := \text{Lnil} : \forall A : \text{Set}, \text{Length} A (\text{nil} A) \text{O}, \text{Lcons} : \forall A : \text{Set}, \forall a : A, \forall l : (\text{List} A), \forall n : \text{nat}, (\text{Length} A l n) \rightarrow (\text{Length} A (\text{cons} A a l) (\text{S} n))\right)
\text{Coq} < \text{Inductive} \text{tree} : \text{Set} := \text{node} : \text{forest} \rightarrow \text{tree}, \text{emptyf} : \text{forest}, \text{consf} : \text{tree} \rightarrow \text{forest} \rightarrow \text{forest}.
\]

Coq Reference Manual, V8.2pl2, June 29, 2010
The Coq type-checker verifies that all parameters are applied in the correct manner in the conclusion of the type of each constructors:

In particular, the following definition will not be accepted because there is an occurrence of \texttt{List} which is not applied to the parameter variable in the conclusion of the type of \texttt{cons'}:

\begin{verbatim}
Coq < Inductive list' (A:Set) : Set :=
Coq < | nil' : list' A
Coq < Coq < Error: Last occurrence of "list'" must have "A" as 1st argument in "A -> list' A -> list' (A * A)%type".
\end{verbatim}

Since Coq version 8.1, there is no restriction about parameters in the types of arguments of constructors. The following definition is valid:

\begin{verbatim}
Coq < Inductive list' (A:Set) : Set :=
Coq < | nil' : list' A
Coq < | cons' : A -> list' (A->A) -> list' A.
list' is defined
list'_rect is defined
list'_ind is defined
list'_rec is defined
\end{verbatim}

4.5.2 Types of inductive objects

We have to give the type of constants in an environment \( E \) which contains an inductive declaration.

\textbf{Ind-Const} Assuming \( \Gamma_I \) is \( [I_1 : A_1; \ldots ; I_k : A_k] \), and \( \Gamma_C \) is \( [c_1 : C_1; \ldots ; c_n : C_n] \),

\[ \text{Ind}(\Gamma)[p](\Gamma_I := \Gamma_C) \in E \implies \begin{cases} 
(I_j : A_j) \in E & j = 1, \ldots , k \ \\
(c_i : C_i) \in E & i = 1, \ldots , n \end{cases} \]

\textbf{Example.} We have \((\text{List} : \text{Set} \to \text{Set}), (\text{cons} : \forall A : \text{Set}, A \to (\text{List} A) \to (\text{List} A)),\) \((\text{Length} : \forall A : \text{Set}, (\text{List} A) \to \text{nat} \to \text{Prop}), \text{tree} : \text{Set} \text{ and forest} : \text{Set}.)\)

From now on, we write \texttt{List\_A} instead of \((\text{List} A)\) and \texttt{Length\_A} for \((\text{Length} A)\).

4.5.3 Well-formed inductive definitions

We cannot accept any inductive declaration because some of them lead to inconsistent systems. We restrict ourselves to definitions which satisfy a syntactic criterion of positivity. Before giving the formal rules, we need a few definitions:

\textbf{Definitions} A type \( T \) is an \textit{arity of sort} \( s \) if it converts to the sort \( s \) or to a product \( \forall x : T, U \) with \( U \) an arity of sort \( s \). (For instance \( A \to \text{Set} \) or \( \forall A : \text{Prop}, A \to \text{Prop} \) are arities of sort respectively \( \text{Set} \) and \( \text{Prop} \). A type of \textit{constructor} of \( I \) is either a term \((I t_1 \ldots t_n)\) or \( \forall x : T, C \) with \( C \) recursively a type of constructor of \( I \).

The type of constructor \( T \) will be said to \textit{satisfy the positivity condition} for a constant \( X \) in the following cases:
The constant $X$ occurs strictly positively in $T$ in the following cases:

- $X$ does not occur in $T$
- $T$ converts to $(X \, t_1 \ldots \, t_n)$ and $X$ does not occur in any $t_i$
- $T$ converts to $\forall \, x : U, V$ and $X$ does not occur in type $V$
- $T$ converts to $(I \, a_1 \ldots \, a_m \, t_1 \ldots \, t_p)$ where $I$ is the name of an inductive declaration of the form $\text{Ind}(\Gamma)[m](I : A := c_1 : \forall p_1 : P_1, \ldots, \forall p_m : P_m, C_1, \ldots, c_n : \forall p_1 : P_1, \ldots, \forall p_m : P_m, C_n)$ (in particular, it is not mutually defined and it has $m$ parameters) and $X$ does not occur in any of the $t_i$, and the (instantiated) types of constructor $C_j\{p_j/a_j\}_j=1..m$ of $I$ satisfy the nested positivity condition for $X$

The type of constructor $T$ of $I$ satisfies the nested positivity condition for a constant $X$ in the following cases:

- $T = (I \, b_1 \ldots \, b_m \, u_1 \ldots \, u_p)$, $I$ is an inductive definition with $m$ parameters and $X$ does not occur in any $u_i$
- $T = \forall \, x : U, V$ and $X$ occurs only strictly positively in $U$ and the type $V$ satisfies the nested positivity condition for $X$

**Example** $X$ occurs strictly positively in $A \rightarrow X$ or $X \ast A$ or $(\text{list} \, X)$ but not in $X \rightarrow A$ or $(X \rightarrow A) \rightarrow A$ nor $(\neg \, X)$ assuming the notion of product and lists were already defined and $\neg$ is an inductive definition with declaration $\text{Ind}(\Gamma)[m](I : A := c_1 : \forall p_1 : P_1, \ldots, \forall p_m : P_m, C_1, \ldots, c_n : \forall p_1 : P_1, \ldots, \forall p_m : P_m, C_n)$ assuming $X$ has arity $\text{nat} \rightarrow \text{Prop}$ and $\text{ex}$ is the inductively defined existential quantifier, the occurrence of $X$ in $(\text{ex} \, \text{nat} \, \lambda \, n : \text{nat}, (X \, n))$ is also strictly positive.

**Correctness rules.** We shall now describe the rules allowing the introduction of a new inductive definition.

**W-Ind** Let $E$ be an environment and $\Gamma, \Gamma_P, \Gamma_I, \Gamma_C$ are contexts such that $\Gamma_I$ is $[I_1 : \forall \Gamma_P, A_1; \ldots; I_k : \forall \Gamma_P, A_k]$ and $\Gamma_C$ is $[c_1 : \forall \Gamma_P, C_1; \ldots; c_n : \forall \Gamma_P, C_n]$.

$$
\begin{align*}
E(\Gamma; \Gamma_P) & \vdash A_j : s'_j)_{j=1\ldots k} (E(\Gamma; \Gamma_P) & \vdash C_i : s_{q_i})_{i=1\ldots n} \\
\forall \forall (\forall (E; \text{Ind}(\Gamma)[p](\Gamma_I := \Gamma_C))[\Gamma]
\end{align*}
$$

provided that the following side conditions hold:

- $k > 0$ and all of $I_j$ and $c_i$ are distinct names for $j = 1 \ldots k$ and $i = 1 \ldots n$,
- $p$ is the number of parameters of $\text{Ind}(\Gamma)(\Gamma_I := \Gamma_C)$ and $\Gamma_P$ is the context of parameters,
- for $j = 1 \ldots k$ we have that $A_j$ is an arity of sort $s_j$ and $I_j \notin \Gamma \cup E$,
- for $i = 1 \ldots n$ we have that $C_i$ is a type of constructor of $I_{q_i}$ which satisfies the positivity condition for $I_1 \ldots I_k$ and $c_i \notin \Gamma \cup E$. 

One can remark that there is a constraint between the sort of the arity of the inductive type and the sort of the type of its constructors which will always be satisfied for the impredicative sort (Prop) but may fail to define inductive definition on sort Set and generate constraints between universes for inductive definitions in the Type hierarchy.

**Examples.** It is well known that existential quantifier can be encoded as an inductive definition. The following declaration introduces the second-order existential quantifier $\exists X. P(X)$.

```coq
Coq < Inductive exProp (P:Type->Prop) : Type
Coq < := exP_intro : forall X:Prop, P X -> exProp P.
```

The same definition on Set is not allowed and fails:

```coq
Coq < := exS_intro : forall X:Set, P X -> exSet P.
Coq < Inductive exSet (P:Set->Prop) : Set

Coq < := exP_intro : forall X:Prop, P X -> exProp P.
Coq < Inductive exProp (P:Prop->Prop) : Prop
```

It is possible to declare the same inductive definition in the universe Type. The `exType` inductive definition has type $(\forall X. P X) \rightarrow Type$ with the constraint that the parameter $X$ of `exT_intro` has type $Type_k$ with $k < j$ and $k \leq i$.

```coq
Coq < := exT_intro : forall X:Type, P X -> exType P.
```

**Sort-polymorphism of inductive families.** From Coq version 8.1, inductive families declared in Type are polymorphic over their arguments in Type.

If $A$ is an arity and $s$ a sort, we write $A/s$ for the arity obtained from $A$ by replacing its sort with $s$. Especially, if $A$ is well-typed in some environment and context, then $A/s$ is typable by typability of all products in the Calculus of Inductive Constructions. The following typing rule is added to the theory.

**Ind-Family** Let $\text{Ind}(\Gamma[p](\Gamma_I := \Gamma_C))$ be an inductive definition. Let $\Gamma_P = [p_1 : P_1; \ldots; p_p : P_p]$ be its context of parameters, $\Gamma_I = [I_1 : \forall \Gamma_P, A_1; \ldots; I_k : \forall \Gamma_P, A_k]$ its context of definitions and $\Gamma_C = [c_1 : \forall \Gamma_P, C_1; \ldots; c_m : \forall \Gamma_P, C_m]$ its context of constructors, with $c_i$ a constructor of $I_{q_i}$.

Let $m \leq l$ be the length of the longest prefix of parameters such that the first $m$ arguments are recursively uniform parameters and the $p - m$ remaining parameters are recursively non-uniform parameters. Let $q_1, \ldots, q_m$ with $0 \leq r \leq m$, be a (possibly) partial instantiation of the recursively uniform parameters of $\Gamma_P$. We have:

$$
\begin{align*}
\text{Ind}(\Gamma[p](\Gamma_I := \Gamma_C)) & \in E \\
(E[\Gamma] \vdash q_1 : P_1')_{l=1\ldots r} \\
(E[\Gamma] \vdash P'_1 \leq \beta_\delta\zeta P_1\{p_u/q_u\}_{u=1\ldots l-1})_{l=1\ldots r} \\
1 \leq j \leq k
\end{align*}
$$

provided that the following side conditions hold:

- $\Gamma_{P'}$ is the context obtained from $\Gamma_P$ by replacing each $P_l$ that is an arity with $P'_l$ for $1 \leq l \leq r$ (notice that $P_l$ arity implies $P'_l$ arity since $E[\Gamma] \vdash P'_l \leq \beta_\delta\zeta P_1\{p_u/q_u\}_{u=1\ldots l-1}$);
• there are sorts \( s_i \), for \( 1 \leq i \leq k \) such that, for \( \Gamma_I = [I_1 : \forall \Gamma P', (A_1)_{s_1}; \ldots ; I_k : \forall \Gamma P', (A_k)_{s_k}] \) we have \((E[\Gamma; \Gamma P'; \Gamma P'] \vdash C_i : s_{q_i})_{i=1 \ldots n}\);

• the sorts are such that all eliminations, to \( \text{Prop} \), \( \text{Set} \) and \( \text{Type}(j) \), are allowed (see section 4.5.4).

Notice that if \( I_j q_1 \ldots q_r \) is typable using the rules \texttt{Ind-Const} and \texttt{App}, then it is typable using the rule \texttt{Ind-Family}. Conversely, the extended theory is not stronger than the theory without \texttt{Ind-Family}. We get an equiconsistency result by mapping each \( \text{Ind}(\Gamma)[p](\Gamma_I := \Gamma C) \) occurring into a given derivation into as many different inductive types and constructors as the number of different (partial) replacements of sorts, needed for this derivation, in the parameters that are arities (this is possible because \( \text{Ind}(\Gamma)[p](\Gamma_I := \Gamma C) \) well-formed implies that \( \text{Ind}(\Gamma)[p](\Gamma_{I'} := \Gamma_{C'}) \) is well-formed and has the same allowed eliminations, where \( \Gamma_{I'} \) is defined as above and \( \Gamma_{C'} = [c_1 : \forall \Gamma P', C_1; \ldots ; c_n : \forall \Gamma P', C_n] \)). That is, the changes in the types of each partial instance \( q_1 \ldots q_r \) can be characterized by the ordered sets of arity sorts among the types of parameters, and to each signature is associated a new inductive definition with fresh names. Conversion is preserved as any (partial) instance \( I_j q_1 \ldots q_r \) or \( C_i q_1 \ldots q_r \) is mapped to the names chosen in the specific instance of \( \text{Ind}(\Gamma)[p](\Gamma_I := \Gamma C) \).

In practice, the rule \texttt{Ind-Family} is used by Coq only when all the inductive types of the inductive definition are declared with an arity whose sort is in the \texttt{Type} hierarchy. Then, the polymorphism is over the parameters whose type is an arity of sort in the \texttt{Type} hierarchy. The sort \( s_j \) are chosen canonically so that each \( s_j \) is minimal with respect to the hierarchy \( \text{Prop} \subset \text{Set} \subset \text{Type} \) where \( \text{Set}_p \) is predicative \texttt{Set}. More precisely, an empty or small singleton inductive definition (i.e. an inductive definition of which all inductive types are singleton – see paragraph 4.5.4) is set in \texttt{Prop}, a small non-singleton inductive family is set in \texttt{Set} (even in case \texttt{Set} is impredicative – see Section 4.7), and otherwise in the \texttt{Type} hierarchy.

Note that the side-condition about allowed elimination sorts in the rule \texttt{Ind-Family} is just to avoid to recompute the allowed elimination sorts at each instance of a pattern-matching (see section 4.5.4).

As an example, let us consider the following definition:

```
Coq < Inductive option (A:Type) : Type :=
  | None : option A
  | Some : A -> option A.
```

As the definition is set in the \texttt{Type} hierarchy, it is used polymorphically over its parameters whose types are arities of a sort in the \texttt{Type} hierarchy. Here, the parameter \( A \) has this property, hence, if \texttt{option} is applied to a type in \texttt{Set}, the result is in \texttt{Set}. Note that if \texttt{option} is applied to a type in \texttt{Prop}, then, the result is not set in \texttt{Prop} but in \texttt{Set} still. This is because \texttt{option} is not a singleton type (see section 4.5.4) and it would loose the elimination to \texttt{Set} and \texttt{Type} if set in \texttt{Prop}.

```
Coq < Check (fun A:Set => option A).
fun A : Set => option A
  : Set -> Set
```

```
Coq < Check (fun A:Prop => option A).
fun A : Prop => option A
  : Prop -> Set
```

Here is another example.

```
Coq < Inductive prod (A B:Type) : Type := pair : A -> B -> prod A B.
```
As \texttt{prod} is a singleton type, it will be in \texttt{Prop} if applied twice to propositions, in \texttt{Set} if applied twice to at least one type in \texttt{Set} and none in \texttt{Type}, and in \texttt{Type} otherwise. In all cases, the three kind of eliminations schemes are allowed.

\begin{verbatim}
Coq < Check (fun A:Set => prod A).
  fun A : Set => prod A                      : Set -> Type -> Type
Coq < Check (fun A:Prop => prod A A).
  fun A : Prop => prod A A                 : Prop -> Prop
Coq < Check (fun (A:Prop) (B:Set) => prod A B).
  fun (A : Prop) (B : Set) => prod A B    : Prop -> Set -> Set
Coq < Check (fun (A:Type) (B:Prop) => prod A B).
  fun (A : Type) (B : Prop) => prod A B   : Type -> Prop -> Type
\end{verbatim}

### 4.5.4 Destructors

The specification of inductive definitions with arities and constructors is quite natural. But we still have to say how to use an object in an inductive type.

This problem is rather delicate. There are actually several different ways to do that. Some of them are logically equivalent but not always equivalent from the computational point of view or from the user point of view.

From the computational point of view, we want to be able to define a function whose domain is an inductively defined type by using a combination of case analysis over the possible constructors of the object and recursion.

Because we need to keep a consistent theory and also we prefer to keep a strongly normalizing reduction, we cannot accept any sort of recursion (even terminating). So the basic idea is to restrict ourselves to primitive recursive functions and functionals.

For instance, assuming a parameter \(A : \) \texttt{Set} exists in the context, we want to build a function \texttt{length} of type \texttt{List} \(\rightarrow\) \texttt{nat} which computes the length of the list, so such that (\texttt{length} \(\texttt{nil}\)) = \texttt{O} and (\texttt{length} \(\texttt{cons}\ \texttt{a}\ \texttt{l}\)) = (\texttt{S} (\texttt{length} \texttt{l})). We want these equalities to be recognized implicitly and taken into account in the conversion rule.

From the logical point of view, we have built a type family by giving a set of constructors. We want to capture the fact that we do not have any other way to build an object in this type. So when trying to prove a property \(P\ m\) for \(m\) in an inductive definition it is enough to enumerate all the cases where \(m\) starts with a different constructor.

In case the inductive definition is effectively a recursive one, we want to capture the extra property that we have built the smallest fixed point of this recursive equation. This says that we are only manipulating finite objects. This analysis provides induction principles.

For instance, in order to prove \(\forall l : \texttt{List} \ A, (\texttt{Length} \ A l \ (\texttt{length} l))\) it is enough to prove: (\texttt{Length} \ A (\texttt{nil} A) \ (\texttt{length} (\texttt{nil} A))) and

\[ \forall a : A, \forall l : \texttt{List} \ A, (\texttt{Length} \ A l \ (\texttt{length} l)) \rightarrow (\texttt{Length} \ A (\texttt{cons} A a l) \ (\texttt{length} (\texttt{cons} A a l))). \]

which given the conversion equalities satisfied by \texttt{length} is the same as proving: (\texttt{Length} \ A (\texttt{nil} A) \texttt{O}) and \(\forall a : A, \forall l : \texttt{List} \ A, (\texttt{Length} \ A l \ (\texttt{length} l)) \rightarrow (\texttt{Length} \ A (\texttt{cons} A a l) \ (\texttt{S} (\texttt{length} l))).\)

One conceptually simple way to do that, following the basic scheme proposed by Martin-Löf in his Intuitionistic Type Theory, is to introduce for each inductive definition an elimination operator. At the
logical level it is a proof of the usual induction principle and at the computational level it implements a
generic operator for doing primitive recursion over the structure.

But this operator is rather tedious to implement and use. We choose in this version of Coq to
factorize the operator for primitive recursion into two more primitive operations as was first suggested
by Th. Coquand in [27]. One is the definition by pattern-matching. The second one is a definition by
guarded fixpoints.

The \textbf{match...with ...end} construction.

The basic idea of this destructor operation is that we have an object \( m \) in an inductive type \( I \) and we
want to prove a property \((P \, m)\) which in general depends on \( m \). For this, it is enough to prove the
property for \( m = (c_i \, u_1 \ldots u_p) \) for each constructor of \( I \).

The Coq term for this proof will be written :

\[
\text{match } m \text{ with } (c_1 \, x_11 \ldots x_{1p_1}) \Rightarrow f_1 \mid \cdots \mid (c_n \, x_{n1}\ldots x_{np_n}) \Rightarrow f_n \text{ end}
\]

In this expression, if \( m \) is a term built from a constructor \((c_i \, u_1 \ldots u_p)\) then the expression will behave
as it is specified with \( i \)-th branch and will reduce to \( f_i \) where the \( x_1\ldots x_{ip_i} \) are replaced by the \( u_1 \ldots u_p \)
according to the \( i \)-reduction.

Actually, for type-checking a \texttt{match...with...end} expression we also need to know the predicate \( P \) to be proved by case analysis. In the general case where \( I \) is an inductively defined \( n \)-ary relation, \( P \) is a \( n + 1 \)-ary relation: the \( n \) first arguments correspond to the arguments of \( I \) (parameters excluded),
and the last one corresponds to object \( m \). Coq can sometimes infer this predicate but sometimes not.
The concrete syntax for describing this predicate uses the \texttt{as...in...return} construction. For instance, let us assume that \( I \) is an unary predicate with one parameter. The predicate is made explicit using the syntax :

\[
\text{match } m \text{ as } x \text{ in } I \_ a \text{ return } (P \, x) \text{ with } (c_1 \, x_11 \ldots x_{1p_1}) \Rightarrow f_1 \mid \cdots \mid (c_n \, x_{n1}\ldots x_{np_n}) \Rightarrow f_n \text{ end}
\]

The \texttt{as} part can be omitted if either the result type does not depend on \( m \) (non-dependent elimination) or
\( m \) is a variable (in this case, the result type can depend on \( m \)). The \texttt{in} part can be omitted if the result type
does not depend on the arguments of \( I \). Note that the arguments of \( I \) corresponding to parameters must
be \( _\_ \), because the result type is not generalized to all possible values of the parameters. The expression
after \texttt{in} must be seen as an inductive type pattern. As a final remark, expansion of implicit arguments
and notations apply to this pattern.

For the purpose of presenting the inference rules, we use a more compact notation :

\[
\text{case}(m, (\lambda ax, P), \lambda x_{11} \ldots x_{1p_1}, f_1 \mid \cdots \mid \lambda x_{n1}\ldots x_{np_n}, f_n)
\]

Allowed elimination sorts. An important question for building the typing rule for \texttt{match} is what can be
the type of \( P \) with respect to the type of the inductive definitions.

We define now a relation \([I : A|B]\) between an inductive definition \( I \) of type \( A \) and an arity \( B \). This
relation states that an object in the inductive definition \( I \) can be eliminated for proving a property \( P \) of
type \( B \).

The case of inductive definitions in sorts \texttt{Set} or \texttt{Type} is simple. There is no restriction on the sort of the
predicate to be eliminated.

Coq Reference Manual, V8.2pl2, June 29, 2010
Notations. The \([I : A|B]\) is defined as the smallest relation satisfying the following rules: We write \([I|B]\) for \([I : A|B]\) where \(A\) is the type of \(I\).

**Prod**

\[
\begin{align*}
(I x) : A'|B' \\
I : (x : A)A'|(x : A)B'
\end{align*}
\]

**Set& Type**

\[
\begin{align*}
s_1 &\in \{\text{Set, Type(j)}\}, s_2 \in S \\
[I : s_1|I \rightarrow s_2]
\end{align*}
\]

The case of Inductive definitions of sort Prop is a bit more complicated, because of our interpretation of this sort. The only harmless allowed elimination, is the one when predicate \(P\) is also of sort Prop.

**Prop**

\[
[I : \text{Prop}|I \rightarrow \text{Prop}]
\]

Prop is the type of logical propositions, the proofs of properties \(P\) in Prop could not be used for computation and are consequently ignored by the extraction mechanism. Assume \(A\) and \(B\) are two propositions, and the logical disjunction \(A \lor B\) is defined inductively by:

\[
\begin{align*}
\text{Coq} &\text{ < Inductive or (A B:Prop) : Prop :=} \\
\text{Coq} &\text{ lintro : A \rightarrow or A B | rintro : B \rightarrow or A B.}
\end{align*}
\]

The following definition which computes a boolean value by case over the proof of or A B is not accepted:

\[
\begin{align*}
\text{Coq} &\text{ < Definition choice (A B: Prop) (x:or A B) :=} \\
\text{Coq} &\text{ match x with lintro a => true | rintro b => false end.}
\end{align*}
\]

From the computational point of view, the structure of the proof of \((or A B)\) in this term is needed for computing the boolean value.

In general, if \(I\) has type Prop then \(P\) cannot have type \(I \rightarrow \text{Set}\), because it will mean to build an informative proof of type \((P m)\) doing a case analysis over a non-computational object that will disappear in the extracted program. But the other way is safe with respect to our interpretation we can have \(I\) a computational object and \(P\) a non-computational one, it just corresponds to proving a logical property of a computational object.

In the same spirit, elimination on \(P\) of type \(I \rightarrow \text{Type}\) cannot be allowed because it trivially implies the elimination on \(P\) of type \(I \rightarrow \text{Set}\) by cumulativity. It also implies that there is two proofs of the same property which are provably different, contradicting the proof-irrelevance property which is sometimes a useful axiom:

\[
\text{Coq < Axiom proof_irrelevance : forall (P : Prop) (x y : P), x=y. proof_irrelevance is assumed}
\]

The elimination of an inductive definition of type Prop on a predicate \(P\) of type \(I \rightarrow \text{Type}\) leads to a paradox when applied to impredicative inductive definition like the second-order existential quantifier exProp defined above, because it give access to the two projections on this type.
Empty and singleton elimination. There are special inductive definitions in \( \text{Prop} \) for which more eliminations are allowed.

**Prop-extended**

\[
I \text{ is an empty or singleton definition } \quad s \in S \quad [I : \text{Prop}|I \rightarrow s]
\]

A \textit{singleton definition} has only one constructor and all the arguments of this constructor have type \( \text{Prop} \). In that case, there is a canonical way to interpret the informative extraction on an object in that type, such that the elimination on any sort \( s \) is legal. Typical examples are the conjunction of non-informative propositions and the equality. If there is an hypothesis \( h : a = b \) in the context, it can be used for rewriting not only in logical propositions but also in any type.

Coq < Print eq_rec.
\[
eq \text{eq_rec} = \text{fun (A : Type) (x : A) (P : A \rightarrow \text{Set}) \Rightarrow eq_rect x P}
\]
\[
: \forall (A : \text{Type}) (x : A) (P : A \rightarrow \text{Set}),
\]
\[
P x \rightarrow \forall y : A, x = y \rightarrow P y
\]
Argument \( A \) is implicit
Argument scopes are \{\text{type_scope } _ _ _ _ _\}

Coq < Extraction eq_rec.
\[
(** \text{val eq_rec : 'a1 \rightarrow 'a2 \rightarrow 'a1 \rightarrow 'a2 **)}
\]
let eq_rec x f y =
\[
f
\]
An empty definition has no constructors, in that case also, elimination on any sort is allowed.

**Type of branches.** Let \( c \) be a term of type \( C \), we assume \( C \) is a type of constructor for an inductive definition \( I \). Let \( P \) be a term that represents the property to be proved. We assume \( r \) is the number of parameters.

We define a new type \( \{c : C\}^P \) which represents the type of the branch corresponding to the \( c : C \) constructor.

\[
\{c : (I, p_1 \ldots p_r t_1 \ldots t_p)\}^P \equiv (P t_1 \ldots t_p c)
\]
\[
\{c : \forall x : T, C\}^P \equiv \forall x : T, \{(c x) : C\}^P
\]

We write \( \{c\}^P \) for \( \{c : C\}^P \) with \( C \) the type of \( c \).

**Examples.** For \( \text{List}_A \) the type of \( P \) will be \( \text{List}_A \rightarrow s \) for \( s \in S \).

\[
\{(\text{cons} A)\}^P \equiv \forall a : A, \forall l : \text{List}_A, (P (\text{cons} A a l)).
\]

For \( \text{Length}_A \), the type of \( P \) will be \( \forall l : \text{List}_A, \forall n : \text{nat}, (\text{Length}_A l n) \rightarrow \text{Prop} \) and the expression \( \{(\text{Length}_A A)\}^P \) is defined as:

\[
\forall a : A, \forall l : \text{List}_A, \forall n : \text{nat}, \forall h : (\text{Length}_A l n), (P (\text{cons} A a l) (S n) (\text{Lcons} A a l n l)).
\]

If \( P \) does not depend on its third argument, we find the more natural expression:

\[
\forall a : A, \forall l : \text{List}_A, \forall n : \text{nat}, (\text{Length}_A l n) \rightarrow (P (\text{cons} A a l) (S n)).
\]

**Typing rule.** Our very general destructor for inductive definition enjoys the following typing rule

\[
\text{match}
\]
\[
E[\Gamma] \vdash c : (I q_1 \ldots q_r t_1 \ldots t_s) \quad E[\Gamma] \vdash P : B \quad [(I q_1 \ldots q_r)|B] \ (E[\Gamma] \vdash f_i : \{(c_{p_i}, q_1 \ldots q_r)\}^P)_{i=1 \ldots l}
\]
\[
E[\Gamma] \vdash \text{case}(c, P, f_1| \ldots | f_l) : (P t_1 \ldots t_s c)
\]
provided \( I \) is an inductive type in a declaration \( \text{Ind}(\Delta)[r](\Gamma_I := \Gamma_C) \) with \( \Gamma_C = [c_1 : C_1; \ldots; c_n : C_n] \) and \( c_{p_1} \ldots c_{p_l} \) are the only constructors of \( I \).

**Example.** For \texttt{List} and \texttt{Length} the typing rules for the \texttt{match} expression are (writing just \( t : M \) instead of \( E[\Gamma] \vdash t : M \), the environment and context being the same in all the judgments).

\[
\begin{align*}
l : \texttt{List}_A & \quad P : \texttt{List}_A \rightarrow s \quad f_1 : (P \texttt{nil} A) \quad f_2 : \forall a : A, \forall l : \texttt{List}_A, (P \texttt{cons} A a l) \\
\text{case}(l, P, f_1 | f_2) & : (P l)
\end{align*}
\]

\[
\begin{align*}
H : (\texttt{Length}_A L N) \\
P : \forall l : \texttt{List}_A, \forall n : \texttt{nat}, \texttt{(Length}_A l n) & \rightarrow \texttt{Prop} \\
f_1 : (P \texttt{nil} A) \land \texttt{nil} & \\
f_2 : \forall a : A, \forall l : \texttt{List}_A, \forall n : \texttt{nat}, \forall (\texttt{Length}_A l n), (P \texttt{cons} A a n) (S n) (\texttt{Lcons} A a l n h) \\
\text{case}(H, P, f_1 | f_2) & : (P L N H)
\end{align*}
\]

**Definition of \( \iota \)-reduction.** We still have to define the \( \iota \)-reduction in the general case. A \( \iota \)-redex is a term of the following form:

\[
\text{case}((c_{p_i} q_1 \ldots q_r a_1 \ldots a_m), P; f_1 | \ldots | f_l)
\]

with \( c_{p_i} \) the \( i \)-th constructor of the inductive type \( I \) with \( r \) parameters.

The \( \iota \)-contraction of this term is \( (f_i a_1 \ldots a_m) \) leading to the general reduction rule:

\[
\text{case}((c_{p_i} q_1 \ldots q_r a_1 \ldots a_m), P; f_1 | \ldots | f_l) \triangleright (f_i a_1 \ldots a_m)
\]

### 4.5.5 Fixpoint Definitions

The second operator for elimination is fixpoint definition. This fixpoint may involve several mutually recursive definitions. The basic concrete syntax for a recursive set of mutually recursive declarations is (with \( \Gamma_i \) contexts):

\[
\text{fix } f_i(\Gamma_i) : A_i := t_i \text{ with } \ldots \text{ with } f_n(\Gamma_n) : A_n := t_n
\]

The terms are obtained by projections from this set of declarations and are written

\[
\text{fix } f_i(\Gamma_i) : A_i := t_i \text{ with } \ldots \text{ with } f_n(\Gamma_n) : A_n := t_n \text{ for } f_i
\]

In the inference rules, we represent such a term by

\[
\text{Fix } f_i\{f_1 : A'_1 := t'_1; \ldots; f_n : A'_n := t'_n\}
\]

with \( t'_i \) (resp. \( A'_i \)) representing the term \( t_i \) abstracted (resp. generalized) with respect to the bindings in the context \( \Gamma_i \), namely \( t'_i = \lambda \Gamma_i, t_i \) and \( A'_i = \forall \Gamma_i, A_i \).
Typing rule

The typing rule is the expected one for a fixpoint.

\[
\frac{(E[\Gamma] \vdash A_i : s_i)_{i=1..n} \quad (E[\Gamma, f_1 : A_1, \ldots, f_n : A_n] \vdash t_i : A_i)_{i=1..n}}{E[\Gamma] \vdash \exists f \{ f_1 : A_1 := t_1 \ldots f_n : A_n := t_n \} : A}
\]

Any fixpoint definition cannot be accepted because non-normalizing terms will lead to proofs of absurdity.

The basic scheme of recursion that should be allowed is the one needed for defining primitive recursive functionals. In that case the fixpoint enjoys a special syntactic restriction, namely one of the arguments belongs to an inductive type, the function starts with a case analysis and recursive calls are done on variables coming from patterns and representing subterms.

For instance in the case of natural numbers, a proof of the induction principle of type

\[
\forall P : \text{nat} \to \text{Prop}, (P \circ 0) \to (\forall n : \text{nat}, (P n) \to (P (S n))) \to \forall n : \text{nat}, (P n)
\]

can be represented by the term:

\[
\lambda P : \text{nat} \to \text{Prop}, \lambda f : (P \circ 0), \lambda g : (\forall n : \text{nat}, (P n) \to (P (S n))), \exists x h\{ h : \forall n : \text{nat}, (P n) := \lambda n : \text{nat}, \text{case}(n, P, f | \lambda p : \text{nat}, (g p (h p))) \}
\]

Before accepting a fixpoint definition as being correctly typed, we check that the definition is "guarded". A precise analysis of this notion can be found in [65].

The first stage is to precise on which argument the fixpoint will be decreasing. The type of this argument should be an inductive definition.

For doing this the syntax of fixpoints is extended and becomes

\[
\exists f \{ f_1/k_1 : A_1 := t_1 \ldots f_n/k_n : A_n := t_n \}
\]

where \( k_i \) are positive integers. Each \( A_i \) should be a type (reducible to a term) starting with at least \( k_i \) products \( \forall y_1 : B_1, \ldots \forall y_k : B_k, A' \) and \( B_k \) being an instance of an inductive definition.

Now in the definition \( t_i \), if \( f_j \) occurs then it should be applied to at least \( k_j \) arguments and the \( k_j \)-th argument should be syntactically recognized as structurally smaller than \( y_k \).

The definition of being structurally smaller is a bit technical. One needs first to define the notion of recursive arguments of a constructor. For an inductive definition \( \text{Ind}(\Gamma)[r](\Gamma_I := \Gamma_C) \), the type of a constructor \( c \) has the form \( \forall p_1 : P_1, \ldots \forall p_r : P_r, \forall x_1 : T_1, \ldots \forall x_r : T_r, (I_j p_1 \ldots p_r t_1 \ldots t_s) \) the recursive arguments will correspond to \( T_j \) in which one of the \( I_j \) occurs.

The main rules for being structurally smaller are the following:

Given a variable \( y \) of type an inductive definition in a declaration \( \text{Ind}(\Gamma)[r](\Gamma_I := \Gamma_C) \), where \( \Gamma_I \) is \( [I_1 : A_1; \ldots; I_k : A_k] \), and \( \Gamma_C \) is \( [c_1 : C_1; \ldots; c_n : C_n] \). The terms structurally smaller than \( y \) are:

- \( (t \ u) \), \( \lambda x : u, t \) when \( t \) is structurally smaller than \( y \).
- \( \text{case}(c, P, f_1 \ldots f_n) \) when each \( f_i \) is structurally smaller than \( y \).

If \( c \) is \( y \) or is structurally smaller than \( y \), its type is an inductive definition \( I_p \) part of the inductive declaration corresponding to \( y \). Each \( f_i \) corresponds to a type of constructor \( C_q \equiv \forall p_1 : P_1, \ldots \forall p_r : P_r, \forall y_1 : B_1, \ldots \forall y_k : B_k, (I_{a_1 \ldots a_k}) \) and can consequently be written \( \lambda y_1 : B'_1, \ldots \lambda y_k : B'_k, g_r \). \( (B'_i) \) is obtained from \( B_i \) by substituting parameters variables \( y_j \) occurring in \( y_i \) corresponding to recursive arguments \( B_i \) (the ones in which one of the \( I_i \) occurs) are structurally smaller than \( y \).
The following definitions are correct, we enter them using the `Fixpoint` command as described in Section 1.3.4 and show the internal representation.

```coq
Coq < Fixpoint plus (n m:nat) {struct n} : nat :=
Coq <    match n with
Coq <      | O => m
Coq <      | S p => S (plus p m)
Coq <    end.

plus is recursively defined (decreasing on 1st argument)

Coq < Fixpoint lgth (A:Set) (l:list A) {struct l} : nat :=
Coq <    match l with
Coq <      | nil => O
Coq <      | cons a l' => S (lgth A l')
Coq <    end.

lgth is recursively defined (decreasing on 2nd argument)

Coq < Fixpoint sizet (t:tree) : nat := let (f) := t in S (sizef f)
Coq < with sizef (f:forest) : nat :=
Coq <    match f with
Coq <      | emptyf => O
Coq <      | consf t f0 => plus (sizet t) (sizef f0)
Coq <    end.

do and sizef are recursively defined (decreasing respectively on 1st, 1st arguments)

Coq < Fixpoint sizet (t:tree) : nat :=
Coq <    let (f) := t in S (sizef f)

Coq < Fixpoint sizet (t:tree) : nat :=
Coq <    let (f) := t in S (sizef f)
```

Argument scopes are `[type_scope_]`

```coq
Coq < Print sizet.

sizet =

fix sizet (t : tree) : nat :=
  let (f) := t in S (sizef f)
with sizef (f : forest) : nat :=
  match f with
  | emptyf => O
  | consf t f0 => plus (sizet t) (sizef f0)
  end
end
for sizet
  : tree -> nat
```
Reduction rule

Let $F$ be the set of declarations: $f_1/k_1 : A_1 := t_1 \ldots f_n/k_n : A_n := t_n$. The reduction for fixpoints is:

$$(\text{Fix } f_1 \{ F \} a_1 \ldots a_{k_1}) \leadsto t_1, t_i \{ (f_k/\text{Fix } f_k \{ F \})_{k=1..n} \} a_1 \ldots a_{k_i}$$

when $a_{k_i}$ starts with a constructor. This last restriction is needed in order to keep strong normalization and corresponds to the reduction for primitive recursive operators.

We can illustrate this behavior on examples.

Coq < Goal forall n m:nat, plus (S n) m = S (plus n m).
1 subgoal

==========
forall n m : nat, plus (S n) m = S (plus n m)

Coq < reflexivity.
Proof completed.
Coq < Abort.
Current goal aborted

Coq < Goal forall f:forest, sizet (node f) = S (sizef f).
1 subgoal

==========
forall f : forest, sizet (node f) = S (sizef f)

Coq < reflexivity.
Proof completed.
Coq < Abort.
Current goal aborted

But assuming the definition of a son function from tree to forest:

Coq < Definition sont (t:tree) : forest
Coq < := let (f) := t in f.
sont is defined

The following is not a conversion but can be proved after a case analysis.

Coq < Goal forall t:tree, sizet t = S (sizef (sont t)).
Coq < Coq < 1 subgoal

==========
forall t : tree, sizet t = S (sizef (sont t))

Coq < reflexivity. (** this one fails **) Toplevel input, characters 0-11:
> reflexivity.
> ^^^^^^^^^^^^ Error: Impossible to unify "S (sizef (sont t))" with "sizet t".

Coq < destruct t.
1 subgoal

f : forest

==========
sizet (node f) = S (sizef (sont (node f)))

Coq < reflexivity.
Proof completed.
4.6 Coinductive types

The implementation contains also coinductive definitions, which are types inhabited by infinite objects. More information on coinductive definitions can be found in [66, 67, 68].

4.7 CIC: the Calculus of Inductive Construction with impredicative Set

COQ can be used as a type-checker for CIC, the original Calculus of Inductive Constructions with an impredicative sort $\text{Set}$ by using the compiler option `-impredicative-set`. For example, using the ordinary `coqtop` command, the following is rejected.

```
Coq < Definition id: Set := forall X:Set,X->X.
Coq < Coq < Coq < Coq < Toplevel input, characters 185-202:
> Definition id: Set := forall X:Set,X->X.
> ^^^^^^^^^^^^^^^^^
Error: The term "forall X : Set, X -> X" has type "Type"
while it is expected to have type "Set".
```

while it will type-check, if one use instead the `coqtop -impredicative-set` command.

The major change in the theory concerns the rule for product formation in the sort $\text{Set}$, which is extended to a domain in any sort $\text{Set}$:

\[
\frac{E[\Gamma] \vdash T : s \quad s \in S \quad E[\Gamma :: (x : T)] \vdash U : \text{Set}}{E[\Gamma] \vdash \forall x : T, U : \text{Set}}
\]

This extension has consequences on the inductive definitions which are allowed. In the impredicative system, one can build so-called large inductive definitions like the example of second-order existential quantifier (`exSet`).

There should be restrictions on the eliminations which can be performed on such definitions. The eliminations rules in the impredicative system for sort $\text{Set}$ become:

\[
\begin{align*}
\frac{s \in \{\text{Prop}, \text{Set}\}}{[I : \text{Set}|I \rightarrow s]} & \quad \text{I is a small inductive definition} \\
\frac{s \in \{\text{Type}(i)\}}{[I : \text{Set}|I \rightarrow s]} & \quad \text{I is a small inductive definition}
\end{align*}
\]
Chapter 5

The Module System

The module system extends the Calculus of Inductive Constructions providing a convenient way to structure large developments as well as a mean of massive abstraction.

5.1 Modules and module types

Access path. It is denoted by \( p \), it can be either a module variable \( X \) or, if \( p' \) is an access path and \( id \) an identifier, then \( p'.id \) is an access path.

Structure element. It is denoted by \( e \) and is either a definition of a constant, an assumption, a definition of an inductive, a definition of a module, an alias of module or a module type abbreviation.

Structure expression. It is denoted by \( S \) and can be:

- an access path \( p \)
- a plain structure \( \text{Struct} \; e; \ldots; e \; \text{End} \)
- a functor \( \text{Functor}(X : S) \; S' \), where \( X \) is a module variable, \( S \) and \( S' \) are structure expression
- an application \( S \; p \), where \( S \) is a structure expression and \( p \) an access path
- a refined structure \( S \) with \( p := p' \) or \( S \) with \( p := t : T \) where \( S \) is a structure expression, \( p \) and \( p' \) are access paths, \( t \) is a term and \( T \) is the type of \( t \).

The symbol \( W \) will be used to denote a plain structure or a functor.

Module definition, is written \( \text{Mod}(X : S \{ := S' \}) \) and consists of a module variable \( X \), a module type \( S \) which can be any structure expression and optionally a module implementation \( S' \) which can be any structure expression except a refined structure.

Module alias, is written \( \text{ModA}(X == p) \) and consists of a module variable \( X \) and a module path \( p \).

Module type abbreviation, is written \( \text{ModType}(Y := S) \), where \( Y \) is an identifier and \( S \) is any structure expression.
5.2 Typing Modules

In order to introduce the typing system we first slightly extend the syntactic class of terms and environments given in section 4.1. The environments, apart from definitions of constants and inductive types now also hold any other structure elements. Terms, apart from variables, constants and complex terms, include also access paths.

We also need additional typing judgments:

- \( E \vdash WF(S) \), denoting that a structure \( S \) is well-formed,
- \( E \vdash p : S \), denoting that the module pointed by \( p \) has type \( S \) in environment \( E \).
- \( E \vdash S \rightarrow W \), denoting that a structure \( S \) is evaluated to a structure \( W \) in weak head normal form.
- \( E \vdash S_1 <: S_2 \), denoting that a structure \( S_1 \) is a subtype of a structure \( S_2 \).
- \( E \vdash e_1 <: e_2 \), denoting that a structure element \( e_1 \) is more precise that a structure element \( e_2 \).

The rules for forming structures are the following:

**WF-STR**

\[
WF(E; E')[] \quad \frac{\vdash WF(S)}{\vdash WF(\text{Struct } E' \text{ End})}
\]

**WF-FUN**

\[
\frac{E; \text{Mod}(X : S)[] \vdash S' \rightarrow W \quad E; \text{Mod}(X : S)[] \vdash WF(W)}{\vdash WF(\text{Functor}(X : S) S')}
\]

Evaluation of structures to weak head normal form:

**WEVAL-APP**

\[
\frac{E[] \vdash S \rightarrow \text{Functor}(X : S_1) S_2 \quad E[] \vdash S_1 \rightarrow W_1 \quad E[] \vdash p : W_3 \quad E[] \vdash W_3 < : W_1}{E[] \vdash Sp \rightarrow S_2(p/X, t_1/p_1.e_1, \ldots, t_n/p_n.e_n)}
\]

In the last rule, \( \{t_1/p_1.e_1, \ldots, t_n/p_n.e_n\} \) is the resulting substitution from the inlining mechanism. We substitute in \( S \) the inlined fields \( p_i.e_i \) form \( \text{Mod}(X : S_1) \) by the corresponding delta-reduced term \( t_i \) in \( p \).

**WEVAL-WITH-MOD**

\[
\frac{E[] \vdash S \rightarrow \text{Struct } e_1; \ldots; e_i-1; \text{Mod}(X : S_1); e_i+2; \ldots; e_n \text{ End} \quad E[] \vdash e_1; \ldots; e_i-1[] \vdash S_1 \rightarrow W_1 \quad E[] \vdash p : W_2 \quad E[] \vdash W_2 < : W_1}{E[] \vdash S \text{ with } x := p \rightarrow \text{Struct } e_1; \ldots; e_i-1; \text{ModA}(X_1 == p); e_i+2\{p/X\}; \ldots; e_n\{p/X\} \text{ End}}
\]

**WEVAL-WITH-MOD-REC**

\[
\frac{E[] \vdash S \rightarrow \text{Struct } e_1; \ldots; e_i-1; \text{Mod}(X_1 : S_1); e_i+2; \ldots; e_n \text{ End} \quad E[] \vdash e_1; \ldots; e_i-1[] \vdash S_1 \text{ with } p := p_1 \rightarrow W_1}{E[] \vdash S \text{ with } X_1.p := p_1 \rightarrow \text{Struct } e_1; \ldots; e_i-1; \text{Mod}(X_1 : W_1); e_i+2\{p_1/X_1.p\}; \ldots; e_n\{p_1/X_1.p\} \text{ End}}
\]
5.2 Typing Modules

\textbf{WEVAL-WITH-DEF}

\[
\begin{align*}
E[] & \vdash S \rightarrow \text{Struct } e_1\ldots; e_{i-1}; \text{Assum}(c : T_1); e_{i+2}\ldots; e_n \text{ End} \\
E; e_1\ldots; e_{i-1} & \vdash \text{Def}(c := t : T) <: \text{Assum}(c : T_1) \\
E[] & \vdash S \text{ with } c := t : T \rightarrow \text{Struct } e_1\ldots; e_{i-1}; \text{Def}(c := t : T); e_{i+2}\ldots; e_n \text{ End}
\end{align*}
\]

\textbf{WEVAL-WITH-DEF-REC}

\[
\begin{align*}
E[] & \vdash S \rightarrow \text{Struct } e_1\ldots; e_{i-1}; \text{Assum}(c : T_1); e_{i+2}\ldots; e_n \text{ End} \\
E; e_1\ldots; e_{i-1} & \vdash \text{Mod}(X_1 : S_1); e_{i+2}\ldots; e_n \text{ End}
\end{align*}
\]

\textbf{WEVAL-PATH-MOD}

\[
\begin{align*}
E[] & \vdash p \rightarrow \text{Struct } e_1\ldots; e_{i-1}; \text{Mod}(X : S \{:= S_1\}); e_{i+2}\ldots; e_n \text{ End} \\
E; e_1\ldots; e_{i-1} & \vdash S \rightarrow W \\
E[] & \vdash p.X \rightarrow W \\
\mathcal{WF}(E[]) & \quad \text{Mod}(X : S \{:= S_1\}) \in E \\
E[] & \vdash S \rightarrow W \\
E[] & \vdash X \rightarrow W
\end{align*}
\]

\textbf{WEVAL-PATH-ALIAS}

\[
\begin{align*}
E[] & \vdash p \rightarrow \text{Struct } e_1\ldots; e_{i-1}; \text{ModA}(X := p_1); e_{i+2}\ldots; e_n \text{ End} \\
E; e_1\ldots; e_{i-1} & \vdash p_1 \rightarrow W \\
E[] & \vdash p.X \rightarrow W \\
\mathcal{WF}(E[]) & \quad \text{ModA}(X := p_1) \in E \\
E[] & \vdash p_1 \rightarrow W \\
E[] & \vdash X \rightarrow W
\end{align*}
\]

\textbf{WEVAL-PATH-TYPE}

\[
\begin{align*}
E[] & \vdash p \rightarrow \text{Struct } e_1\ldots; e_{i-1}; \text{ModType}(Y := S); e_{i+2}\ldots; e_n \text{ End} \\
E; e_1\ldots; e_{i-1} & \vdash S \rightarrow W \\
E[] & \vdash p.Y \rightarrow W \\
\mathcal{WF}(E[]) & \quad \text{ModType}(Y := S) \in E \\
E[] & \vdash S \rightarrow W \\
E[] & \vdash Y \rightarrow W
\end{align*}
\]

Rules for typing module:

\textbf{MT-EVAL-STR}

\[
\begin{align*}
E[] & \vdash p \rightarrow W \\
E[] & \vdash p : W/p
\end{align*}
\]
The notation $W/p$ has the following meaning:

- if $W \rightarrow \text{Struct } e_1; \ldots; e_n$ End then $W/p = \text{Struct } e_1/p; \ldots; e_n/p$ End where $e/p$ is defined as follows:
  - $\text{Def}(c := t : T)/p^! = \text{Def}(c := t : T)$
  - $\text{Assum}(c : U)/p = \text{Def}(c := p.c : U)$
  - $\text{Mod}(X : S)/p = \text{ModA}(X == p.X)$
  - $\text{ModA}(X == p')/p = \text{ModA}(X == p')$
  - $\text{Ind}()[\Gamma_P] [\Gamma_C := \Gamma_I]/p = \text{Ind}_p()[\Gamma_P] [\Gamma_C := \Gamma_I]$
  - $\text{Ind}_p()[\Gamma_P] [\Gamma_C := \Gamma_I]/p = \text{Ind}_p()[\Gamma_P] [\Gamma_C := \Gamma_I]$

- if $W \rightarrow \text{Functor}(X : S') S''$ then $W/p = W$

The notation $\text{Ind}_p()[\Gamma_P] [\Gamma_C := \Gamma_I]$ denotes an inductive definition that is definitionally equal to the inductive definition in the module denoted by the path $p$. All rules which have $\text{Ind}_p()[\Gamma_P] [\Gamma_C := \Gamma_I]$ as premises are also valid for $\text{Ind}_p()[\Gamma_P] [\Gamma_C := \Gamma_I]$. We give the formation rule for $\text{Ind}_p()[\Gamma_P] [\Gamma_C := \Gamma_I]$ below as well as the equality rules on inductive types and constructors.

The module subtyping rules:

**MSUB-STR**

$E\{e_1;\ldots; e_n\} \vdash e_{\sigma(i)} < : e'_i$ for $i = 1..m$

$\sigma : \{1\ldots m\} \rightarrow \{1\ldots n\}$ injective

$E\{\} \vdash \text{Struct } e_1; \ldots; e_n$ End $< : \text{Struct } e'_1; \ldots; e'_m$ End

**MSUB-FUN**

$E\{\} \vdash S_1 \rightarrow W_1$

$E\{\} \vdash S'_1 \rightarrow W'_1$

$E; \text{Mod}(X : S_1)\{\} \vdash S_2 \rightarrow W_2$

$E; \text{Mod}(X : S'_1)\{\} \vdash S'_2 \rightarrow W'_2$

$E\{\} \vdash W'_1 < : W_1$

$E; \text{Mod}(X : S'_1)\{\} \vdash W_2 < : W'_2$

$E\{\} \vdash \text{Functor}(X : S_1) S_2 < : \text{Functor}(X : S'_1) S'_2$

Structure element subtyping rules:

**ASSUM-ASSUM**

$E\{\} \vdash T_1 \leq_{\beta\delta\iota\zeta} T_2$

$E\{\} \vdash \text{Assum}(c : T_1) < : \text{Assum}(c : T_2)$

**DEF-ASSUM**

$E\{\} \vdash T_1 \leq_{\beta\delta\iota\zeta} T_2$

$E\{\} \vdash \text{Def}(c := t : T_1) < : \text{Assum}(c : T_2)$

**ASSUM-DEF**

$E\{\} \vdash T_1 \leq_{\beta\delta\iota\zeta} T_2$

$E\{\} \vdash c =_{\beta\delta\iota\zeta} t_2$

$E\{\} \vdash \text{Assum}(c : T_1) < : \text{Def}(c := t_2 : T_2)$

1Opaque definitions are processed as assumptions.
5.2 Typing Modules

\[
\begin{align*}
\text{DEF-DEF} & \quad E[\vDash T_1 \leq \beta \delta \zeta T_2, \quad E[\vDash t_1 = \beta \delta \zeta t_2] \quad \Rightarrow \quad E[\vDash \text{Def}(c := t_1 : T_1) < : \text{Def}(c := t_2 : T_2)] \\
\text{IND-IND} & \quad E[\vDash \Gamma_P = \beta \delta \zeta \Gamma' P, \quad E[\Gamma_P] \vDash \Gamma_C = \beta \delta \zeta \Gamma'_C, \quad E[\Gamma_P; \Gamma_C] \vDash \Gamma_I = \beta \delta \zeta \Gamma'_I] \quad \Rightarrow \quad E[\vDash \text{Ind}(\Gamma_P)(\Gamma_C := \Gamma_I) < : \text{Ind}(\Gamma'_P)(\Gamma'_C := \Gamma'_I)] \\
\text{INDP-IND} & \quad E[\vDash \Gamma_P = \beta \delta \zeta \Gamma' P, \quad E[\Gamma_P] \vDash \Gamma_C = \beta \delta \zeta \Gamma'_C, \quad E[\Gamma_P; \Gamma_C] \vDash \Gamma_I = \beta \delta \zeta \Gamma'_I] \quad \Rightarrow \quad E[\vDash \text{Ind}_{p}(\Gamma_P)(\Gamma_C := \Gamma_I) < : \text{Ind}_{p'}(\Gamma'_P)(\Gamma'_C := \Gamma'_I)] \\
\text{INDP-INDP} & \quad E[\vDash \Gamma_P = \beta \delta \zeta \Gamma' P, \quad E[\Gamma_P] \vDash \Gamma_C = \beta \delta \zeta \Gamma'_C, \quad E[\Gamma_P; \Gamma_C] \vDash \Gamma_I = \beta \delta \zeta \Gamma'_I, \quad E[\vDash p = \beta \delta \zeta p'] \quad \Rightarrow \quad E[\vDash \text{Ind}_{p}(\Gamma_P)(\Gamma_C := \Gamma_I) < : \text{Ind}_{p'}(\Gamma'_P)(\Gamma'_C := \Gamma'_I)] \\
\text{MOD-MOD} & \quad E[\vDash S_1 < : S_2] \quad \Rightarrow \quad E[\vDash \text{Mod}(X : S_1) < : \text{Mod}(X : S_2)] \\
\text{ALIAS-MOD} & \quad E[\vDash p : S_1] \quad E[\vDash S_1 < : S_2] \quad \Rightarrow \quad E[\vDash \text{Mod}(X == p) < : \text{Mod}(X : S_2)] \\
\text{MOD-ALIAS} & \quad E[\vDash p : S_2] \quad E[\vDash S_2 < : S_1] \quad E[\vDash X = \beta \delta \zeta p] \quad \Rightarrow \quad E[\vDash \text{Mod}(X : S_1) < : \text{Mod}(X == p)] \\
\text{ALIAS-ALIAS} & \quad E[\vDash p_1 = \beta \delta \zeta p_2] \quad \Rightarrow \quad E[\vDash \text{Mod}(X == p_1) < : \text{Mod}(X == p_2)] \\
\text{MODTYPE-MODTYPE} & \quad E[\vDash S_1 < : S_2] \quad E[\vDash S_2 < : S_1] \quad \Rightarrow \quad E[\vDash \text{ModType}(Y := S_1) < : \text{ModType}(Y := S_2)]
\end{align*}
\]

New environment formation rules

\[
\begin{align*}
\text{WF-MOD} & \quad \text{WF}(E)[] \quad E[\vDash \text{WF}(S)] \quad \Rightarrow \quad \text{WF}(E; \text{Mod}(X : S))[] \\
\text{WF-MOD} & \quad E[\vDash S_2 < : S_1] \quad \text{WF}(E)[] \quad E[\vDash \text{WF}(S_1)] \quad E[\vDash \text{WF}(S_2)] \quad \Rightarrow \quad \text{WF}(E; \text{Mod}(X : S_1 I := S_2 I))[] \\
\text{WF-ALIAS} & \quad \text{WF}(E)[] \quad E[\vDash p : S] \quad \Rightarrow \quad \text{WF}(E, \text{Mod}(X == p))[]
\end{align*}
\]
WF-MODTYPE

\[
\begin{align*}
\text{WF}(E)[[]] &\vdash \text{WF}(S)[[]] \\
\text{WF}(E, \text{ModType}(Y := S))[[]] 
\end{align*}
\]

WF-IND

\[
\begin{align*}
\text{WF}(E; \text{Ind}()[\Gamma_p][\Gamma_C := \Gamma_I])[]
\end{align*}
\]

\[
\begin{align*}
E[] \vdash p : \text{Struct}\ e_1; \ldots; e_i; \text{Assum}() (c : T); \ldots\ \text{End} \\
E[] \vdash p.c : T
\end{align*}
\]

\[
\begin{align*}
E[] \vdash p : \text{Struct}\ e_1; \ldots; e_i; \text{Def}() (c := t : T); \ldots\ \text{End} \\
E[] \vdash p.c : T
\end{align*}
\]

Component access rules

ACC-TYPE

\[
\begin{align*}
E[\Gamma] \vdash p : \text{Struct}\ e_1; \ldots; e_i; \text{Assum}() (c : T); \ldots\ \text{End} \\
E[\Gamma] \vdash p.c : T
\end{align*}
\]

ACC-Delta

Notice that the following rule extends the delta rule defined in section 4.3

\[
\begin{align*}
E[\Gamma] \vdash p : \text{Struct}\ e_1; \ldots; e_i; \text{Def}() (c := t : U); \ldots\ \text{End} \\
E[\Gamma] \vdash p.c \vdash t
\end{align*}
\]

In the rules below we assume \(\Gamma_P\) is \([p_1 : P_1; \ldots; p_r : P_r]\), \(\Gamma_I\) is \([I_1 : A_1; \ldots; I_k : A_k]\), and \(\Gamma_C\) is \([c_1 : C_1; \ldots; c_n : C_n]\)

ACC-IND

\[
\begin{align*}
E[\Gamma] \vdash p : \text{Struct}\ e_1; \ldots; e_i; \text{Ind}() [\Gamma_P][\Gamma_C := \Gamma_I]; \ldots\ \text{End} \\
E[\Gamma] \vdash p.I_j : (p_1 : P_1) \ldots (p_r : P_r)A_j
\end{align*}
\]

\[
\begin{align*}
E[\Gamma] \vdash p : \text{Struct}\ e_1; \ldots; e_i; \text{Ind}() [\Gamma_P][\Gamma_C := \Gamma_I]; \ldots\ \text{End} \\
E[\Gamma] \vdash p.c_m : (p_1 : P_1) \ldots (p_r : P_r)C_m I_j (I_j p_1 \ldots p_r)_{j=1\ldots k}
\end{align*}
\]

ACC-INDP

\[
\begin{align*}
E[[]] \vdash p : \text{Struct}\ e_1; \ldots; e_i; \text{Ind}_p() [\Gamma_P][\Gamma_C := \Gamma_I]; \ldots\ \text{End} \\
E[[]] \vdash p.I_i \vdash \delta.p'.I_i
\end{align*}
\]

\[
\begin{align*}
E[[]] \vdash p : \text{Struct}\ e_1; \ldots; e_i; \text{Ind}_p() [\Gamma_P][\Gamma_C := \Gamma_I]; \ldots\ \text{End} \\
E[[]] \vdash p.e_i \vdash \delta.p'.e_i
\end{align*}
\]
Part II

The proof engine
Chapter 6

Vernacular commands

6.1 Displaying

6.1.1 Print \textit{qualid}.

This command displays on the screen informations about the declared or defined object referred by \textit{qualid}.

Error messages:

1. \textit{qualid} not a defined object

Variants:

1. Print Term \textit{qualid}.
   This is a synonym to Print \textit{qualid} when \textit{qualid} denotes a global constant.

2. About \textit{qualid}.
   This displays various informations about the object denoted by \textit{qualid}: its kind (module, constant, assumption, inductive, constructor, abbreviation...), long name, type, implicit arguments and argument scopes. It does not print the body of definitions or proofs.

6.1.2 Print All.

This command displays informations about the current state of the environment, including sections and modules.

Variants:

1. Inspect \textit{num}.
   This command displays the \textit{num} last objects of the current environment, including sections and modules.

2. Print Section \textit{ident}.
   should correspond to a currently open section, this command displays the objects defined since the beginning of this section.
6.2 Requests to the environment

6.2.1 Check term.

This command displays the type of term. When called in proof mode, the term is checked in the local context of the current subgoal.

6.2.2 Eval convtactic in term.

This command performs the specified reduction on term, and displays the resulting term with its type. The term to be reduced may depend on hypothesis introduced in the first subgoal (if a proof is in progress).

See also: Section 8.5.

6.2.3 Compute term.

This command performs a call-by-value evaluation of term by using the bytecode-based virtual machine. It is a shortcut for Eval vm_compute in term.

See also: Section 8.5.

6.2.4 Extraction term.

This command displays the extracted term from term. The extraction is processed according to the distinction between Set and Prop; that is to say, between logical and computational content (see Section 4.1.1). The extracted term is displayed in Objective Caml syntax, where global identifiers are still displayed as in COQ terms.

Variants:

1. Recursive Extraction qualid₁ ... qualidₙ.
   Recursively extracts all the material needed for the extraction of globals qualid₁ ... qualidₙ.

See also: Chapter 21.

6.2.5 Print Assumptions qualid.

This commands display all the assumptions (axioms, parameters and variables) a theorem or definition depends on. Especially, it informs on the assumptions with respect to which the validity of a theorem relies.

6.2.6 Search qualid.

This command displays the name and type of all theorems of the current context whose statement’s conclusion has the form (qualid t₁ . . . tₙ). This command is useful to remind the user of the name of library lemmas. Error messages:

1. The reference qualid was not found in the current environment
   There is no constant in the environment named qualid.
6.2 Requests to the environment

Variants:

1. **Search** `qualid inside module_1 ... module_n`.
   This restricts the search to constructions defined in modules `module_1 ... module_n`.

2. **Search** `qualid outside module_1 ... module_n`.
   This restricts the search to constructions not defined in modules `module_1 ... module_n`.

Error messages:

(a) Module/section `module` not found No module `module` has been required (see Section 6.4.1).

6.2.7 **SearchAbout** `qualid`.

This command displays the name and type of all objects (theorems, axioms, etc) of the current context whose statement contains `qualid`. This command is useful to remind the user of the name of library lemmas.

Error messages:

1. The reference `qualid` was not found in the current environment
   There is no constant in the environment named `qualid`.

Variants:

1. **SearchAbout** `string`.
   If `string` is a valid identifier, this command displays the name and type of all objects (theorems, axioms, etc) of the current context whose name contains `string`. If `string` is a notation’s string denoting some reference `qualid` (referred to by its main symbol as in "+" or by its notation’s string as in "_ + _" or "_ 'U' _", see Section 12.1), the command works like **SearchAbout** `qualid`.

2. **SearchAbout** `string%key`.
   The string `string` must be a notation or the main symbol of a notation which is then interpreted in the scope bound to the delimiting key `key` (see Section 12.2.2).

3. **SearchAbout** `term_pattern`.
   This searches for all statements or types of definition that contains a subterm that matches the pattern `term_pattern` (holes of the pattern are either denoted by “_” or by “?ident” when non linear patterns are expected).

4. **SearchAbout** `[ [−]term_pattern-string ... [−]term_pattern-string ]`.
   where `term_pattern-string` is a `term_pattern` or a `string`, or a `string` followed by a scope delimiting key `%key`.

This generalization of **SearchAbout** searches for all objects whose statement or type contains a subterm matching `term_pattern` (or `qualid` if `string` is the notation for a reference `qualid`) and whose name contains all `string` of the request that correspond to valid identifiers. If a `term_pattern` or a `string` is prefixed by “−”, the search excludes the objects that mention that `term_pattern` or that `string`.

Coq Reference Manual, V8.2pl2, June 29, 2010
5. SearchAbout term_pattern-string inside module_1 ... module_n.
   SearchAbout [ term_pattern-string ... term_pattern-string ] inside module_1 ... module_n.
   This restricts the search to constructions defined in modules module_1 ... module_n.

6. SearchAbout term_pattern-string outside module_1 ... module_n.
   SearchAbout [ term_pattern-string ... term_pattern-string ] outside module_1 ... module_n.
   This restricts the search to constructions not defined in modules module_1 ... module_n.

Examples:

Coq < Require Import ZArith.
Coq < SearchAbout [ Zmult Zplus "distr" ].

weak_Zmult_plus_distr_r:  
  forall (p : positive) (n m : Z),  
  (Zpos p * (n + m))%Z = (Zpos p * n + Zpos p * m)%Z
Zmult_plus_distr_r:  
  forall n m p : Z, (n * (m + p))%Z = (n * m + n * p)%Z
Zmult_plus_distr_l:  
  forall n m p : Z, ((n + m) * p)%Z = (n * m + p + m)%Z
fast_Zmult_plus_distr_l:  
  forall (n m p : Z) (P : Z -> Prop),  
  P (n * p + m * p)%Z -> P ((n + m) * p)%Z

Coq < SearchAbout [ "+"%Z "*"%Z "distr" -positive -Prop].
Zmult_plus_distr_r:  
  forall n m p : Z, (n * (m + p))%Z = (n * m + n * p)%Z
Zmult_plus_distr_l:  
  forall n m p : Z, ((n + m) * p)%Z = (n * m + p + m)%Z

Coq < SearchAbout (?x * _ + ?x * _)%Z outside OmegaLemmas.

weak_Zmult_plus_distr_r:  
  forall (p : positive) (n m : Z),  
  (Zpos p * (n + m))%Z = (Zpos p * n + Zpos p * m)%Z
Zmult_plus_distr_r:  
  forall n m p : Z, (n * (m + p))%Z = (n * m + n * p)%Z

6.2.8 SearchPattern term.

This command displays the name and type of all theorems of the current context whose statement’s conclusion matches the expression term where holes in the latter are denoted by “_”. It is a variant of SearchAbout term_pattern that does not look for subterms but searches for statements whose conclusion has exactly the expected form.

Coq < Require Import Arith.
Coq < SearchPattern (_ + _ = _ + _).

plus_comm: forall n m : nat, n + m = m + n
plus_Snm_nSm: forall n m : nat, S n + m = n + S m
plus_assoc: forall n m p : nat, n + (m + p) = n + m + p
plus_permute: forall n m p : nat, n + (m + p) = m + (n + p)
plus_assoc_reverse: forall n m p : nat, n + m + p = n + (m + p)
plus_permute_2_in_4:  
  forall n m p q : nat, n + m + (p + q) = n + p + (m + q)
Patterns need not be linear: you can express that the same expression must occur in two places by using pattern variables ‘?ident’.

Coq < Require Import Arith.
Coq < SearchPattern (?X1 + _ = _ + ?X1).
plus_comm: forall n m : nat, n + m = m + n

Variants:
1. SearchPattern term inside module1 ... module_n.
   This restricts the search to constructions defined in modules module1 ... module_n.

2. SearchPattern term outside module1 ... module_n.
   This restricts the search to constructions not defined in modules module1 ... module_n.

6.2.9 SearchRewrite term.
This command displays the name and type of all theorems of the current context whose statement’s conclusion is an equality of which one side matches the expression term. Holes in term are denoted by “_”.

Coq < Require Import Arith.
Coq < SearchRewrite (_ + _ + _).
plus_assoc: forall n m p : nat, n + (m + p) = n + m + p
plus_assoc_reverse: forall n m p : nat, n + m + p = n + (m + p)
plus_permute_2_in_4:
   forall n m p q : nat, n + m + (p + q) = n + p + (m + q)

Variants:
1. SearchRewrite term inside module1 ... module_n.
   This restricts the search to constructions defined in modules module1 ... module_n.

2. SearchRewrite term outside module1 ... module_n.
   This restricts the search to constructions not defined in modules module1 ... module_n.

6.2.10 Locate qualid.
This command displays the full name of the qualified identifier qualid and consequently the COQ module in which it is defined.

Coq < Locate nat.
Inductive Coq.Init.Datatypes.nat
Coq < Locate Datatypes.O.
Constructor Coq.Init.Datatypes.O (shorter name to refer to it in current context is O)
Coq < Locate Init.Datatypes.O.
Constructor Coq.Init.Datatypes.O
(shorter name to refer to it in current context is O)
Coq < Locate Coq.Init.Datatypes.O.
Constructor Coq.Init.Datatypes.O
  (shorter name to refer to it in current context is O)
Coq < Locate I.Dont.Exist.
No object of suffix I.Dont.Exist

See also: Section 12.1.10

6.2.11 The WHELP searching tool

WHELP is an experimental searching and browsing tool for the whole COQ library and the whole set of COQ user contributions. WHELP requires a browser to work. WHELP has been developed at the University of Bologna as part of the HELM¹ and MoWGLI² projects. It can be invoked directly from the COQ toplevel or from COQIDE, assuming a graphical environment is also running. The browser to use can be selected by setting the environment variable COQREMOTE BROWSER. If not explicitly set, it defaults to firefox -remote "OpenURL(%s,new-tab)" || firefox %s &" or C:\PROGRA~1\INTERN~1\IEXPLORE %s, depending on the underlying operating system (in the command, the string %s serves as metavariable for the url to open). The Whelp tool relies on a dedicated Whelp server and on another server called Getter that retrieves formal documents. The default Whelp server name can be obtained using the command Test Whelp Server and the default Getter can be obtained using the command: Test Whelp Getter. The Whelp server name can be changed using the command:

Set Whelp Server string.
where string is a URL (e.g. http://mowgli.cs.unibo.it:58080).

The Getter can be changed using the command:

Set Whelp Getter string.
where string is a URL (e.g. http://mowgli.cs.unibo.it:58081).

The WHELP commands are:

Whelp Locate "reg_expr".
This command opens a browser window and displays the result of seeking for all names that match the regular expression reg_expr in the COQ library and user contributions. The regular expression can contain the special operators are * and ? that respectively stand for an arbitrary substring and for exactly one character.

Variant: Whelp Locate ident.
This is equivalent to Whelp Locate "ident".

Whelp Match pattern.
This command opens a browser window and displays the result of seeking for all statements that match the pattern pattern. Holes in the pattern are represented by the wildcard character “_”.

¹Hypertextual Electronic Library of Mathematics
²Mathematics on the Web, Get it by Logics and Interfaces
6.3 Loading files

Whelp Instance pattern.

This command opens a browser window and displays the result of seeking for all statements that are instances of the pattern pattern. The pattern is here assumed to be an universally quantified expression.

Whelp Elim qualid.

This command opens a browser window and displays the result of seeking for all statements that have the “form” of an elimination scheme over the type denoted by qualid.

Whelp Hint term.

This command opens a browser window and displays the result of seeking for all statements that can be instantiated so that to prove the statement term.

Variant: Whelp Hint.

This is equivalent to Whelp Hint goal where goal is the current goal to prove. Notice that Coq does not send the local environment of definitions to the WHELP tool so that it only works on requests strictly based on, only, definitions of the standard library and user contributions.

6.3 Loading files

Coq offers the possibility of loading different parts of a whole development stored in separate files. Their contents will be loaded as if they were entered from the keyboard. This means that the loaded files are ASCII files containing sequences of commands for Coq’s toplevel. This kind of file is called a *script* for Coq. The standard (and default) extension of Coq’s script files is .v.

6.3.1 Load ident.

This command loads the file named ident.v, searching successively in each of the directories specified in the loadpath. (see Section 6.5)

Variants:

1. Load string.
   Loads the file denoted by the string string, where string is any complete filename. Then the ~ and .. abbreviations are allowed as well as shell variables. If no extension is specified, Coq will use the default extension .v.

2. Load Verbose ident., Load Verbose string
   Display, while loading, the answers of Coq to each command (including tactics) contained in the loaded file See also: Section 6.8.1

Error messages:

1. Can’t find file ident on loadpath

6.4 Compiled files

This section describes the commands used to load compiled files (see Chapter 13 for documentation on how to compile a file). A compiled file is a particular case of module called *library file.*
6.4.1 Require \textit{qualid}.

This command looks in the loadpath for a file containing module \textit{qualid} and adds the corresponding module to the environment of \textsc{Coq}. As library files have dependencies in other library files, the command \texttt{Require qualid} recursively requires all library files the module \textit{qualid} depends on and adds the corresponding modules to the environment of \textsc{Coq} too. \textsc{Coq} assumes that the compiled files have been produced by a valid \textsc{Coq} compiler and their contents are then not replayed nor rechecked.

To locate the file in the file system, \textit{qualid} is decomposed under the form \texttt{dirpath . ident} and the file \texttt{ident . vo} is searched in the physical directory of the file system that is mapped in \textsc{Coq} loadpath to the logical path \texttt{dirpath} (see Section 6.5). The mapping between physical directories and logical names at the time of requiring the file must be consistent with the mapping used to compile the file.

Variants:

1. \texttt{Require Import qualid}.
   
   This loads and declares the module \textit{qualid} and its dependencies then imports the contents of \textit{qualid} as described in Section 2.5.8.

   It does not import the modules on which \textit{qualid} depends unless these modules were itself required in module \textit{qualid} using \texttt{Require Export}, as described below, or recursively required through a sequence of \texttt{Require Export}.

   If the module required has already been loaded, \texttt{Require Import qualid} simply imports it, as \texttt{Import qualid} would.

2. \texttt{Require Export qualid}.

   This command acts as \texttt{Require Import qualid}, but if a further module, say \textit{A}, contains a command \texttt{Require Export B}, then the command \texttt{Require Import A} also imports the module \textit{B}.

3. \texttt{Require [Import / Export] qualid\textsubscript{1} ... qualid\textsubscript{n}}.

   This loads the modules \textit{qualid\textsubscript{1}}, \ldots, \textit{qualid\textsubscript{n}} and their recursive dependencies. If \texttt{Import} or \texttt{Export} is given, it also imports \textit{qualid\textsubscript{1}}, \ldots, \textit{qualid\textsubscript{n}} and all the recursive dependencies that were marked or transitively marked as \texttt{Export}.

4. \texttt{Require [Import / Export] string}.

   This shortcuts the resolution of the qualified name into a library file name by directly requiring the module to be found in file \texttt{string . vo}.

Error messages:

1. Cannot load \textit{qualid}: no physical path bound to \texttt{dirpath}

2. Cannot find library \textit{foo} in loadpath

   The command did not find the file \texttt{foo . vo}. Either \texttt{foo . v} exists but is not compiled or \texttt{foo . vo} is in a directory which is not in \textsc{your LoadPath} (see Section 6.5).

3. Compiled library \textit{ident . vo} makes inconsistent assumptions over library \textit{qualid}

   The command tried to load library file \texttt{ident . vo} that depends on some specific version of library \textit{qualid} which is not the one already loaded in the current \textsc{Coq} session. Probably \texttt{ident . v} was not properly recompiled with the last version of the file containing module \textit{qualid}. 

4. Bad magic number

The file `ident.vo` was found but either it is not a COQ compiled module, or it was compiled with
an older and incompatible version of COQ.

5. The file `ident.vo` contains library `dirpath` and not library `dirpath’`

The library file `dirpath’` is indirectly required by the Require command but it is bound in the
current loadpath to the file `ident.vo` which was bound to a different library name `dirpath` at the
time it was compiled.

See also: Chapter 13

6.4.2 Print Libraries.

This command displays the list of library files loaded in the current COQ session. For each of these
libraries, it also tells if it is imported.

6.4.3 Declare ML Module `string_1 .. string_n`.

This command loads the Objective Caml compiled files `string_1 .. string_n` (dynamic link). It is mainly
used to load tactics dynamically. The files are searched into the current Objective Caml loadpath (see the
command Add ML Path in the Section 6.5). Loading of Objective Caml files is only possible under
the bytecode version of coqtop (i.e. coqtop called with options -byte, see chapter 13), or when
Coq has been compiled with a version of Objective Caml that supports native Dynlink (≥ 3.11).

Error messages:

1. File not found on loadpath : `string`

2. Loading of ML object file forbidden in a native Coq

6.4.4 Print ML Modules.

This print the name of all OBJECTIVE CAML modules loaded with Declare ML Module. To know
from where these module were loaded, the user should use the command Locate File (see Sec-
tion 6.5.10)

6.5 Loadpath

There are currently two loadpaths in COQ. A loadpath where seeking COQ files (extensions .v or .vo
or .vi) and one where seeking Objective Caml files. The default loadpath contains the directory “.”
denoting the current directory and mapped to the empty logical path (see Section 2.6.2).

6.5.1 Pwd.

This command displays the current working directory.
6.5.2  **Cd**  *string*.

This command changes the current directory according to *string* which can be any valid path.

**Variants:**

1. **Cd**.
   
   Is equivalent to **Pwd**.

6.5.3  **Add LoadPath**  *string*  **as**  *dirpath*.

This command adds the physical directory *string* to the current COQ loadpath and maps it to the logical directory *dirpath*, which means that every file *dirname/basename.v* physically lying in subdirectory *string/dirname* becomes accessible in COQ through absolute logical name *dirpath . dirname . basename*.

**Remark:** **Add LoadPath** also adds *string* to the current ML loadpath.

**Variants:**

1. **Add LoadPath**  *string*.
   
   Performs as **Add LoadPath**  *string*  **as**  *dirpath*  but for the empty directory path.

6.5.4  **Add Rec LoadPath**  *string*  **as**  *dirpath*.

This command adds the physical directory *string* and all its subdirectories to the current COQ loadpath. The top directory *string* is mapped to the logical directory *dirpath* and any subdirectory *pdir* of it is mapped to logical name *dirpath . pdir* and recursively. Subdirectories corresponding to invalid COQ identifiers are skipped, and, by convention, subdirectories named CVS or _darcs are skipped too.

Otherwise, said, **Add Rec LoadPath**  *string*  **as**  *dirpath*  behaves as **Add LoadPath**  *string*  **as**  *dirpath* excepts that files lying in validly named subdirectories of *string* need not be qualified to be found.

In case of files with identical base name, files lying in most recently declared *dirpath* are found first and explicit qualification is required to refer to the other files of same base name.

If several files with identical base name are present in different subdirectories of a recursive loadpath declared via a single instance of **Add Rec LoadPath**, which of these files is found first is system-dependent and explicit qualification is recommended.

**Remark:** **Add Rec LoadPath** also recursively adds *string* to the current ML loadpath.

**Variants:**

1. **Add Rec LoadPath**  *string*.
   
   Works as **Add Rec LoadPath**  *string*  **as**  *dirpath*  but for the empty logical directory path.

6.5.5  **Remove LoadPath**  *string*.

This command removes the path *string* from the current COQ loadpath.
6.6 States and Reset

### 6.5.6 Print LoadPath.

This command displays the current Coq loadpath.

**Variants:**

1. Print LoadPath *dirpath*.
   - Works as Print LoadPath but displays only the paths that extend the *dirpath* prefix.

### 6.5.7 Add ML Path *string*.

This command adds the path *string* to the current Objective Caml loadpath (see the command Declare ML Module in the Section 6.4).

**Remark:** This command is implied by Add LoadPath *string* as *dirpath*.

### 6.5.8 Add Rec ML Path *string*.

This command adds the directory *string* and all its subdirectories to the current Objective Caml loadpath (see the command Declare ML Module in the Section 6.4).

**Remark:** This command is implied by Add Rec LoadPath *string* as *dirpath*.

### 6.5.9 Print ML Path *string*.

This command displays the current Objective Caml loadpath. This command makes sense only under the bytecode version of coqtop, i.e. using option -byte (see the command Declare ML Module in the section 6.4).

### 6.5.10 Locate File *string*.

This command displays the location of file *string* in the current loadpath. Typically, *string* is a .cmo or .vo or .v file.

### 6.5.11 Locate Library *dirpath*.

This command gives the status of the Coq module *dirpath*. It tells if the module is loaded and if not searches in the load path for a module of logical name *dirpath*.

### 6.6 States and Reset

### 6.6.1 Reset *ident*.

This command removes all the objects in the environment since *ident* was introduced, including *ident*. *ident* may be the name of a defined or declared object as well as the name of a section. One cannot reset over the name of a module or of an object inside a module.

**Error messages:**

1. *ident*: no such entry
6.6.2 Back.

This command undoes all the effects of the last vernacular command. This does not include commands that only access to the environment like those described in the previous sections of this chapter (for instance `Require` and `Load` can be undone, but not `Check` and `Locate`). Commands read from a vernacular file are considered as a single command.

Variants:

1. *Back n*

   Undoes *n* vernacular commands.

Error messages:

1. Reached begin of command history

   Happens when there is vernacular command to undo.

6.6.3 Backtrack *num*₁ *num*₂ *num*₃.

This command is dedicated for the use in graphical interfaces. It allows to backtrack to a particular *global* state, i.e. typically a state corresponding to a previous line in a script. A global state includes declaration environment but also proof environment (see Chapter 7). The three numbers *num*₁, *num*₂ and *num*₃ represent the following:

- *num*₃: Number of *Abort* to perform, i.e. the number of currently opened nested proofs that must be canceled (see Chapter 7).
- *num*₂: *Proof state number* to unbury once aborts have been done. Coq will compute the number of *Undo* to perform (see Chapter 7).
- *num*₁: Environment state number to unbury, Coq will compute the number of *Back* to perform.

How to get state numbers?

Notice that when in `-emacs` mode, Coq displays the current proof and environment state numbers in the prompt. More precisely the prompt in `-emacs` mode is the following:

```
<prompt> idᵢ < num₁ | id₁|id₂|...|idₙ | num₂ < </prompt>
```

Where:

- *idᵢ* is the name of the current proof (if there is one, otherwise Coq is displayed, see Chapter 7).
- *num₁* is the environment state number after the last command.
- *num₂* is the proof state number after the last command.
- *id₁ id₂...idₙ* are the currently opened proof names (order not significant).

It is then possible to compute the Backtrack command to unbury the state corresponding to a particular prompt. For example, suppose the current prompt is:

```
< goal4 < 35 | goal1 | goal4 | goal3 | goal2 | | 8 < </prompt>
```

and we want to backtrack to a state labeled by:

```
< goal2 < 32 | goal1 | goal2 | 12 < </prompt>
```
We have to perform `Backtrack 32 12 2`, i.e. perform 2 `Abort`s (to cancel goal4 and goal3), then rewind proof until state 12 and finally go back to environment state 32. Notice that this supposes that proofs are nested in a regular way (no `Resume` or `Suspend` commands).

**Variants:**

1. `BackTo n`
   Is a more basic form of `Backtrack` where only the first argument (global environment number) is given, no `abort` and no `Undo` is performed.

### 6.6.4 Restore State `string`.

Restores the state contained in the file `string`.

**Variants:**

1. `Restore State ident`
   Equivalent to `Restore State "ident.coq"`.

2. `Reset Initial`.
   Goes back to the initial state (like after the command `coqtop`, when the interactive session began). This command is only available interactively.

### 6.6.5 Write State `string`.

Writes the current state into a file `string` for use in a further session. This file can be given as the `inputstate` argument of the commands `coqtop` and `coqc`.

**Variants:**

1. `Write State ident`
   Equivalent to `Write State "ident.coq"`. The state is saved in the current directory (see Section 6.5.1).

### 6.7 Quitting and debugging

#### 6.7.1 Quit.

This command permits to quit COQ.

#### 6.7.2 Drop.

This is used mostly as a debug facility by COQ’s implementors and does not concern the casual user. This command permits to leave COQ temporarily and enter the Objective Caml toplevel. The Objective Caml command:

```
#use "include";;
```

add the right loadpaths and loads some toplevel printers for all abstract types of COQ- `section_path`, `identifiers`, `terms`, `judgments`, …. You can also use the file `base_include` instead, that loads only the pretty-printers for `section_paths` and `identifiers`. You can return back to COQ with the command:
go();;

Warnings:

1. It only works with the bytecode version of COQ (i.e. coqtop called with option -byte, see the contents of Section 13.1).

2. You must have compiled COQ from the source package and set the environment variable COQTOP to the root of your copy of the sources (see Section 13.4).

6.7.3 Time command.
This command executes the vernacular command command and display the time needed to execute it.

6.8 Controlling display

6.8.1 Set Silent.
This command turns off the normal displaying.

6.8.2 Unset Silent.
This command turns the normal display on.

6.8.3 Set Printing Width integer.
This command sets which left-aligned part of the width of the screen is used for display.

6.8.4 Unset Printing Width.
This command resets the width of the screen used for display to its default value (which is 78 at the time of writing this documentation).

6.8.5 Test Printing Width.
This command displays the current screen width used for display.

6.8.6 Set Printing Depth integer.
This command sets the nesting depth of the formatter used for pretty-printing. Beyond this depth, display of subterms is replaced by dots.

6.8.7 Unset Printing Depth.
This command resets the nesting depth of the formatter used for pretty-printing to its default value (at the time of writing this documentation, the default value is 50).

6.8.8 Test Printing Depth.
This command displays the current nesting depth used for display.
6.9 Controlling the reduction strategies and the conversion algorithm

COQ provides reduction strategies that the tactics can invoke and two different algorithms to check the convertibility of types. The first conversion algorithm lazily compares applicative terms while the other is a brute-force but efficient algorithm that first normalizes the terms before comparing them. The second algorithm is based on a bytecode representation of terms similar to the bytecode representation used in the ZINC virtual machine [90]. It is specially useful for intensive computation of algebraic values, such as numbers, and for reflexion-based tactics. The commands to fine-tune the reduction strategies and the lazy conversion algorithm are described first.

6.9.1 Opaque \textit{qualid} \_1 \ldots \textit{qualid} \_n.

This command has an effect on unfoldable constants, i.e. on constants defined by Definition or Let (with an explicit body), or by a command assimilated to a definition such as Fixpoint, Program Definition, etc, or by a proof ended by Defined. The command tells not to unfold the constants \textit{qualid} \_1 \ldots \textit{qualid} \_n in tactics using \(\delta\)-conversion (unfolding a constant is replacing it by its definition).

Opaque has also an effect on the conversion algorithm of COQ, telling to delay the unfolding of a constant as later as possible in case COQ has to check the conversion (see Section 4.3) of two distinct applied constants.

The scope of Opaque is limited to the current section, or current file, unless the variant Global Opaque \textit{qualid} \_1 \ldots \textit{qualid} \_n is used.

See also: sections 8.5, 8.12, 7.1.4

Error messages:

1. The reference \textit{qualid} was not found in the current environment

   There is no constant referred by \textit{qualid} in the environment. Nevertheless, if you asked Opaque foo bar and if bar does not exist, foo is set opaque.

6.9.2 Transparent \textit{qualid} \_1 \ldots \textit{qualid} \_n.

This command is the converse of Opaque and it applies on unfoldable constants to restore their unfoldability after an Opaque command.

Note in particular that constants defined by a proof ended by Qed are not unfoldable and Transparent has no effect on them. This is to keep with the usual mathematical practice of proof irrelevance: what matters in a mathematical development is the sequence of lemma statements, not their actual proofs. This distinguishes lemmas from the usual defined constants, whose actual values are of course relevant in general.

The scope of Transparent is limited to the current section, or current file, unless the variant Global Transparent \textit{qualid} \_1 \ldots \textit{qualid} \_n is used.

Error messages:

1. The reference \textit{qualid} was not found in the current environment

   There is no constant referred by \textit{qualid} in the environment.

See also: sections 8.5, 8.12, 7.1.4
6.9.3 **Strategy level** [ qualid\textsubscript{1} \ldots qualid\textsubscript{n} ].

This command generalizes the behavior of *Opaque* and *Transparent* commands. It is used to fine-tune the strategy for unfolding constants, both at the tactic level and at the kernel level. This command associates a level to qualid\textsubscript{1} \ldots qualid\textsubscript{n}. Whenever two expressions with two distinct head constants are compared (for instance, this comparison can be triggered by a type cast), the one with lower level is expanded first. In case of a tie, the second one (appearing in the cast type) is expanded.

Levels can be one of the following (higher to lower):

**opaque**: level of opaque constants. They cannot be expanded by tactics (behaves like $+\infty$, see next item).

**num**: levels indexed by an integer. Level 0 corresponds to the default behavior, which corresponds to transparent constants. This level can also be referred to as *transparent*. Negative levels correspond to constants to be expanded before normal transparent constants, while positive levels correspond to constants to be expanded after normal transparent constants.

**expand**: level of constants that should be expanded first (behaves like $-\infty$)

These directives survive section and module closure, unless the command is prefixed by *Local*. In the latter case, the behavior regarding sections and modules is the same as for the *Transparent* and *Opaque* commands.

6.9.4 **Set Virtual Machine**

This activates the bytecode-based conversion algorithm.

6.9.5 **Unset Virtual Machine**

This deactivates the bytecode-based conversion algorithm.

6.9.6 **Test Virtual Machine**

This tells if the bytecode-based conversion algorithm is activated. The default behavior is to have the bytecode-based conversion algorithm deactivated.

See also: sections 8.5.1 and 13.5.
Chapter 7

Proof handling

In Coq’s proof editing mode all top-level commands documented in Chapter 6 remain available and the user has access to specialized commands dealing with proof development pragmas documented in this section. He can also use some other specialized commands called tactics. They are the very tools allowing the user to deal with logical reasoning. They are documented in Chapter 8.

When switching in editing proof mode, the prompt Coq < is changed into ident < where ident is the declared name of the theorem currently edited.

At each stage of a proof development, one has a list of goals to prove. Initially, the list consists only in the theorem itself. After having applied some tactics, the list of goals contains the subgoals generated by the tactics.

To each subgoal is associated a number of hypotheses we call the local context of the goal. Initially, the local context is empty. It is enriched by the use of certain tactics (see mainly Section 8.3.5).

When a proof is achieved the message Proof completed is displayed. One can then store this proof as a defined constant in the environment. Because there exists a correspondence between proofs and terms of $\lambda$-calculus, known as the Curry-Howard isomorphism [75, 6, 71, 78], Coq stores proofs as terms of CiC. Those terms are called proof terms.

It is possible to edit several proofs at the same time: see section 7.1.8

Error message: When one attempts to use a proof editing command out of the proof editing mode, Coq raises the error message : No focused proof.

7.1 Switching on/off the proof editing mode

7.1.1 Goal form .

This command switches Coq to editing proof mode and sets form as the original goal. It associates the name Unnamed_thm to that goal.

Error messages:

1. the term form has type ... which should be Set, Prop or Type

See also: Section 7.1.4
7.1.2 Qed.

This command is available in interactive editing proof mode when the proof is completed. Then Qed extracts a proof term from the proof script, switches back to COQ top-level and attaches the extracted proof term to the declared name of the original goal. This name is added to the environment as an Opaque constant.

Error messages:

1. Attempt to save an incomplete proof
2. Sometimes an error occurs when building the proof term, because tactics do not enforce completely the term construction constraints.

The user should also be aware of the fact that since the proof term is completely rechecked at this point, one may have to wait a while when the proof is large. In some exceptional cases one may even incur a memory overflow.

Variants:

1. Defined.
   Defines the proved term as a transparent constant.
2. Save.
   Is equivalent to Qed.
3. Save ident.
   Forces the name of the original goal to be ident. This command (and the following ones) can only be used if the original goal has been opened using the Goal command.
4. Save Theorem ident.
   Save Lemma ident.
   Save Remark ident.
   Save Fact ident.
   Are equivalent to Save ident.

7.1.3 Admitted.

This command is available in interactive editing proof mode to give up the current proof and declare the initial goal as an axiom.

7.1.4 Theorem ident : form.

This command switches to interactive editing proof mode and declares ident as being the name of the original goal form. When declared as a Theorem, the name ident is known at all section levels: Theorem is a global lemma.

Error messages:

1. the term form has type ... which should be Set, Prop or Type
2. **ident already exists**
   
The name you provided already defined. You have then to choose another name.

**Variants:**

1. **Lemma**  
   
   The name you provided already defined. You have then to choose another name.

2. **Remark**  
   
   The name you provided already defined. You have then to choose another name.

3. **Fact**  
   
   The name you provided already defined. You have then to choose another name.

4. **Definition**  
   
   The name you provided already defined. You have then to choose another name.

### 7.1.5 Proof term.

This command applies in proof editing mode. It is equivalent to `exact term; Save`. That is, you have to give the full proof in one gulp, as a proof term (see Section 8.2.1).

**Variant:** Proof.

Is a noop which is useful to delimit the sequence of tactic commands which start a proof, after a Theorem command. It is a good practice to use Proof as an opening parenthesis, closed in the script with a closing Qed.

**See also:** Proof with tactic. in Section 8.13.6.

### 7.1.6 Abort.

This command cancels the current proof development, switching back to the previous proof development, or to the COQ toplevel if no other proof was edited.

**Error messages:**

1. No focused proof (No proof-editing in progress)

**Variants:**

1. **Abort ident.**

   Aborts the editing of the proof named ident.

2. **Abort All.**

   Aborts all current goals, switching back to the COQ toplevel.
7.1.7 Suspend.

This command applies in proof editing mode. It switches back to the COQ toplevel, but without canceling the current proofs.

7.1.8 Resume.

This command switches back to the editing of the last edited proof.

Error messages:

1. No proof-editing in progress

Variants:

1. Resume ident.

Restarts the editing of the proof named ident. This can be used to navigate between currently edited proofs.

Error messages:

1. No such proof

7.1.9 Existential num := term.

This command allows to instantiate an existential variable. num is an index in the list of uninstantiated existential variables displayed by Show Existentials. (described in Section 7.3.1)

This command is intended to be used to instantiate existential variables when the proof is completed but some uninstantiated existential variables remain. To instantiate existential variables during proof edition, you should use the tactic instantiate.

See also: instantiate (num := term). in Section 8.3.15.

7.2 Navigation in the proof tree

7.2.1 Undo.

This command cancels the effect of the last tactic command. Thus, it backtracks one step.

Error messages:

1. No focused proof (No proof-editing in progress)
2. Undo stack would be exhausted

Variants:

1. Undo num.

Repeats Undo num times.
7.3 Requesting information

7.2.2 Set Undo $\text{num}$.

This command changes the maximum number of Undo’s that will be possible when doing a proof. It only affects proofs started after this command, such that if you want to change the current undo limit inside a proof, you should first restart this proof.

7.2.3 Unset Undo.

This command resets the default number of possible Undo commands (which is currently 12).

7.2.4 Restart.

This command restores the proof editing process to the original goal.

Error messages:

1. No focused proof to restart

7.2.5 Focus.

This focuses the attention on the first subgoal to prove and the printing of the other subgoals is suspended until the focused subgoal is solved or unfocused. This is useful when there are many current subgoals which clutter your screen.

Variant:

1. Focus $\text{num}$.

   This focuses the attention on the $\text{num}^{\text{th}}$ subgoal to prove.

7.2.6 Unfocus.

Turns off the focus mode.

7.3 Requesting information

7.3.1 Show.

This command displays the current goals.

Variants:

1. Show $\text{num}$.
   
   Displays only the $\text{num}$-th subgoal.

   Error messages:

   (a) No such goal
   (b) No focused proof

2. Show Implicits.

   Displays the current goals, printing the implicit arguments of constants.
3. Show Implicits $num$.
   Same as above, only displaying the $num$-th subgoal.

4. Show Script.
   Displays the whole list of tactics applied from the beginning of the current proof. This tactics script may contain some holes (subgoals not yet proved). They are printed under the form $<Your Tactic Text here>$. 

5. Show Tree.
   This command can be seen as a more structured way of displaying the state of the proof than that provided by Show Script. Instead of just giving the list of tactics that have been applied, it shows the derivation tree constructed by then. Each node of the tree contains the conclusion of the corresponding sub-derivation (i.e. a goal with its corresponding local context) and the tactic that has generated all the sub-derivations. The leaves of this tree are the goals which still remain to be proved.

   It displays the proof term generated by the tactics that have been applied. If the proof is not completed, this term contain holes, which correspond to the sub-terms which are still to be constructed. These holes appear as a question mark indexed by an integer, and applied to the list of variables in the context, since it may depend on them. The types obtained by abstracting away the context from the type of each hole-placer are also printed.

7. Show Conjectures.
   It prints the list of the names of all the theorems that are currently being proved. As it is possible to start proving a previous lemma during the proof of a theorem, this list may contain several names.

8. Show Intro.
   If the current goal begins by at least one product, this command prints the name of the first product, as it would be generated by an anonymous Intro. The aim of this command is to ease the writing of more robust scripts. For example, with an appropriate Proof General macro, it is possible to transform any anonymous Intro into a qualified one such as Intro $y_1$3. In the case of a non-product goal, it prints nothing.

9. Show Intros.
   This command is similar to the previous one, it simulates the naming process of an Intros.

10. Show Existentials
    It displays the set of all uninstantiated existential variables in the current proof tree, along with the type and the context of each variable.

7.3.2 Guarded.

Some tactics (e.g. refine 8.2.2) allow to build proofs using fixpoint or co-fixpoint constructions. Due to the incremental nature of interactive proof construction, the check of the termination (or guardedness) of the recursive calls in the fixpoint or cofixpoint constructions is postponed to the time of the completion of the proof.

The command Guarded allows to verify if the guard condition for fixpoint and cofixpoint is violated at some time of the construction of the proof without having to wait the completion of the proof.
7.3.3 Set Hyps Limit \textit{num}.

This command sets the maximum number of hypotheses displayed in goals after the application of a tactic. All the hypotheses remains usable in the proof development.

7.3.4 Unset Hyps Limit.

This command goes back to the default mode which is to print all available hypotheses.
Chapter 8

Tactics

A deduction rule is a link between some (unique) formula, that we call the \textit{conclusion} and (several) formulas that we call the \textit{premises}. Indeed, a deduction rule can be read in two ways. The first one has the shape: \textit{“if I know this and this then I can deduce this”}. For instance, if I have a proof of \textit{A} and a proof of \textit{B} then I have a proof of \textit{A \land B}. This is forward reasoning from premises to conclusion. The other way says: \textit{“to prove this I have to prove this and this”}. For instance, to prove \textit{A \land B}, I have to prove \textit{A} and I have to prove \textit{B}. This is backward reasoning which proceeds from conclusion to premises. We say that the conclusion is \textit{the goal} to prove and premises are \textit{the subgoals}. The tactics implement \textit{backward reasoning}. When applied to a goal, a tactic replaces this goal with the subgoals it generates. We say that a tactic reduces a goal to its subgoal(s).

Each (sub)goal is denoted with a number. The current goal is numbered 1. By default, a tactic is applied to the current goal, but one can address a particular goal in the list by writing \textit{n:tactic} which means \textit{“apply tactic tactic to goal number n”}. We can show the list of subgoals by typing \texttt{Show} (see Section 7.3.1).

Since not every rule applies to a given statement, every tactic cannot be used to reduce any goal. In other words, before applying a tactic to a given goal, the system checks that some \textit{preconditions} are satisfied. If it is not the case, the tactic raises an error message.

Tactics are build from atomic tactics and tactic expressions (which extends the folklore notion of tactical) to combine those atomic tactics. This chapter is devoted to atomic tactics. The tactic language will be described in Chapter 9.

There are, at least, three levels of atomic tactics. The simplest one implements basic rules of the logical framework. The second level is the one of \textit{derived rules} which are built by combination of other tactics. The third one implements heuristics or decision procedures to build a complete proof of a goal.

8.1 Invocation of tactics

A tactic is applied as an ordinary command. If the tactic does not address the first subgoal, the command may be preceded by the wished subgoal number as shown below:

\begin{verbatim}
tactic_invocation ::= num : tactic .
| tactic .
\end{verbatim}
8.2 Explicit proof as a term

8.2.1 exact term

This tactic applies to any goal. It gives directly the exact proof term of the goal. Let $T$ be our goal, let $p$ be a term of type $U$ then $\text{exact } p$ succeeds iff $T$ and $U$ are convertible (see Section 4.3).

Error messages:

1. Not an exact proof

Variants:

1. $e\text{exact } term$
   This tactic behaves like $\text{exact}$ but is able to handle terms with meta-variables.

8.2.2 refine term

This tactic allows to give an exact proof but still with some holes. The holes are noted “_”.

Error messages:

1. invalid argument: the tactic $\text{refine}$ doesn’t know what to do with the term you gave.

2. Refine passed ill-formed term: the term you gave is not a valid proof (not easy to debug in general). This message may also occur in higher-level tactics, which call $\text{refine}$ internally.

3. Cannot infer a term for this placeholder there is a hole in the term you gave which type cannot be inferred. Put a cast around it.

An example of use is given in Section 10.1.

8.3 Basics

Tactics presented in this section implement the basic typing rules of $pCIC$ given in Chapter 4.

8.3.1 assumption

This tactic applies to any goal. It implements the “Var” rule given in Section 4.2. It looks in the local context for an hypothesis which type is equal to the goal. If it is the case, the subgoal is proved. Otherwise, it fails.

Error messages:

1. No such assumption

Variants:

1. $e\text{assumption}$
   This tactic behaves like $\text{assumption}$ but is able to handle goals with meta-variables.
8.3 Basics

8.3.2 clear ident

This tactic erases the hypothesis named ident in the local context of the current goal. Then ident is no more displayed and no more usable in the proof development.

Variants:
1. clear ident_1 ... ident_n
   This is equivalent to clear ident_1. ... clear ident_n.
2. clearbody ident
   This tactic expects ident to be a local definition then clears its body. Otherwise said, this tactic turns a definition into an assumption.
3. clear - ident_1 ... ident_n
   This tactic clears all hypotheses except the ones depending in the hypotheses named ident_1 ... ident_n and in the goal.
4. clear
   This tactic clears all hypotheses except the ones depending in goal.

Error messages:
1. ident not found
2. ident is used in the conclusion
3. ident is used in the hypothesis ident'

8.3.3 move ident_1 after ident_2

This moves the hypothesis named ident_1 in the local context after the hypothesis named ident_2.

If ident_1 comes before ident_2 in the order of dependences, then all hypotheses between ident_1 and ident_2 which (possibly indirectly) depend on ident_1 are moved also.

If ident_1 comes after ident_2 in the order of dependences, then all hypotheses between ident_1 and ident_2 which (possibly indirectly) occur in ident_1 are moved also.

Variants:
1. move ident_1 before ident_2
   This moves ident_1 towards and just before the hypothesis named ident_2.
2. move ident at top
   This moves ident at the top of the local context (at the beginning of the context).
3. move ident at bottom
   This moves ident at the bottom of the local context (at the end of the context).

Error messages:
1. ident_i not found
2. Cannot move ident_1 after ident_2: it occurs in ident_2
3. Cannot move ident_1 after ident_2: it depends on ident_2
8.3.4 rename \texttt{ident}_1\ into \texttt{ident}_2

This renames hypothesis \texttt{ident}_1\ into \texttt{ident}_2\ in the current context\footnote{but it does not rename the hypothesis in the proof-term...}

**Variants:**

1. rename \texttt{ident}_1\ into \texttt{ident}_2, \ldots, \texttt{ident}_{2k-1}\ into \texttt{ident}_{2k}

   Is equivalent to the sequence of the corresponding atomic rename.

**Error messages:**

1. \texttt{ident}_2\ not found
2. \texttt{ident}_2\ is already used

8.3.5 intro

This tactic applies to a goal which is either a product or starts with a let binder. If the goal is a product, the tactic implements the “Lam” rule given in Section 4.2\footnote{Actually, only the second subgoal will be generated since the other one can be automatically checked.}. If the goal starts with a let binder then the tactic implements a mix of the “Let” and “Conv”.

If the current goal is a dependent product \texttt{forall}\  \texttt{x:T}, \texttt{U} (resp \texttt{let}\ \texttt{x:=t} in \texttt{U}) then \texttt{intro} puts \texttt{x:T} (resp \texttt{x:=t}) in the local context. The new subgoal is \texttt{U}.

If the goal is a non dependent product \texttt{T \rightarrow U}, then it puts in the local context either \texttt{\forall n:T} (if \texttt{T} is of type \texttt{Set} or \texttt{Prop}) or \texttt{x\ n:T} (if the type of \texttt{T} is \texttt{Type}). The optional index \texttt{n} is such that \texttt{\forall n} or \texttt{x\ n} is a fresh identifier. In both cases the new subgoal is \texttt{U}.

If the goal is neither a product nor starting with a let definition, the tactic \texttt{intro} applies the tactic \texttt{red} until the tactic \texttt{intro} can be applied or the goal is not reducible.

**Error messages:**

1. No product even after head-reduction
2. \texttt{ident} is already used

**Variants:**

1. \texttt{intros}

   Repeats \texttt{intro} until it meets the head-constant. It never reduces head-constants and it never fails.

2. \texttt{intro \ ident}

   Applies \texttt{intro} but forces \texttt{ident} to be the name of the introduced hypothesis.

   **Error message:** \texttt{name ident is already used}

**Remark:** If a name used by \texttt{intro} hides the base name of a global constant then the latter can still be referred to by a qualified name (see 2.6.2).
3. \texttt{intros ident_1 \ldots ident_n}
   Is equivalent to the composed tactic \texttt{intro ident_1; \ldots; intro ident_n}.

More generally, the \texttt{intros} tactic takes a pattern as argument in order to introduce names for components of an inductive definition or to clear introduced hypotheses; This is explained in 8.7.3.

4. \texttt{intros until ident}
   Repeats \texttt{intro} until it meets a premise of the goal having form (ident:term) and discharges the variable named \texttt{ident} of the current goal.

   \textbf{Error message:} No such hypothesis in current goal

5. \texttt{intros until num}
   Repeats \texttt{intro} until the num-th non-dependent product. For instance, on the subgoal \texttt{forall x y:nat, x=y -> y=x} the tactic \texttt{intros until 1} is equivalent to \texttt{intros x y H}, as \texttt{x=y -> y=x} is the first non-dependent product. And on the subgoal \texttt{forall x y z:nat, x=y -> y=x} the tactic \texttt{intros until 1} is equivalent to \texttt{intros x y z} as the product on \texttt{z} can be rewritten as a non-dependent product: \texttt{forall x y:nat, nat -> x=y -> y=x}

   \textbf{Error message:} No such hypothesis in current goal

   Happens when \texttt{num} is 0 or is greater than the number of non-dependent products of the goal.

6. \texttt{intro after ident}
   \texttt{intro before ident}
   \texttt{intro at top}
   \texttt{intro at bottom}

   Applies \texttt{intro} and moves the freshly introduced hypothesis respectively after the hypothesis \texttt{ident}, before the hypothesis \texttt{ident}, at the top of the local context, or at the bottom of the local context. All hypotheses on which the new hypothesis depends are moved too so as to respect the order of dependencies between hypotheses. Note that \texttt{intro at bottom} is a synonym for \texttt{intro} with no argument.

   \textbf{Error messages:}
   
   (a) No product even after head-reduction
   (b) No such hypothesis: \texttt{ident}

7. \texttt{intro ident_1 after ident_2}
   \texttt{intro ident_1 before ident_2}
   \texttt{intro ident_1 at top}
   \texttt{intro ident_1 at bottom}

   Behaves as previously but naming the introduced hypothesis \texttt{ident_1}. It is equivalent to \texttt{intro ident_1} followed by the appropriate call to \texttt{move} (see Section 8.3.3).

   \textbf{Error messages:}
   
   (a) No product even after head-reduction
   (b) No such hypothesis: \texttt{ident}
8.3.6 apply term

This tactic applies to any goal. The argument \textit{term} is a term well-formed in the local context. The tactic \textit{apply} tries to match the current goal against the conclusion of the type of \textit{term}. If it succeeds, then the tactic returns as many subgoals as the number of non dependent premises of the type of \textit{term}. If the conclusion of the type of \textit{term} does not match the goal \textit{and} the conclusion is an inductive type isomorphic to a tuple type, then each component of the tuple is recursively matched to the goal in the left-to-right order.

The tactic \textit{apply} relies on first-order unification with dependent types unless the conclusion of the type of \textit{term} is of the form \((P \ t_1 \ldots t_n)\) with \(P\) to be instantiated. In the latter case, the behavior depends on the form of the goal. If the goal is of the form \((\text{fun } x \Rightarrow Q) \ u_1 \ldots u_n\) and the \(t_i\) and \(u_i\) unifies, then \(P\) is taken to be \((\text{fun } x \Rightarrow Q)\). Otherwise, \textit{apply} tries to define \(P\) by abstracting over \(t_1 \ldots t_n\) in the goal. See \textit{pattern} in Section 8.5.7 to transform the goal so that it gets the form \((\text{fun } x \Rightarrow Q) \ u_1 \ldots u_n\).

\textbf{Error messages:}

1. Impossible to unify \textit{...} with \textit{...}

The \textit{apply} tactic failed to match the conclusion of \textit{term} and the current goal. You can help the \textit{apply} tactic by transforming your goal with the \textit{change} or \textit{pattern} tactics (see sections 8.5.7, 8.3.11).

2. Unable to find an instance for the variables \textit{ident} \ldots \textit{ident}

This occurs when some instantiations of the premises of \textit{term} are not deducible from the unification. This is the case, for instance, when you want to apply a transitivity property. In this case, you have to use one of the variants below:

\textbf{Variants:}

1. apply \textit{term} with \textit{term}_1 \ldots \textit{term}_n

Provides \textit{apply} with explicit instantiations for all dependent premises of the type of \textit{term} which do not occur in the conclusion and consequently cannot be found by unification. Notice that \textit{term}_1 \ldots \textit{term}_n must be given according to the order of these dependent premises of the type of \textit{term}.

\textbf{Error message:} Not the right number of missing arguments

2. apply \textit{term} with (\textit{ref}_1 := \textit{term}_1) \ldots (\textit{ref}_n := \textit{term}_n)

This also provides \textit{apply} with values for instantiating premises. Here, variables are referred by names and non-dependent products by increasing numbers (see syntax in Section 8.3.17).

3. apply \textit{term}_1, \ldots, \textit{term}_n

This is a shortcut for \textit{apply} \textit{term}_1 ; [ .. | .. ]; [ .. | \textit{apply} \textit{term}_n ]; \ldots \], i.e. for the successive applications of \textit{term}_{i+1} on the last subgoal generated by \textit{apply} \textit{term}_i, starting from the application of \textit{term}_1.

4. eapply \textit{term}

The tactic \textit{eapply} behaves as \textit{apply} but does not fail when no instantiation are deducible for some variables in the premises. Rather, it turns these variables into so-called existential variables.
which are variables still to instantiate. An existential variable is identified by a name of the form \(?n\) where \(n\) is a number. The instantiation is intended to be found later in the proof.

An example of use of \texttt{eapply} is given in Section 10.2.

5. \texttt{simple apply term}

This behaves like \texttt{apply} but it reasons modulo conversion only on subterms that contain no variables to instantiate. For instance, if \(\text{id} := \text{fun } x: \text{nat} \rightarrow x\) and \(H : \text{forall } y, \text{id } y = y\) then \texttt{simple apply H} on goal \(O = 0\) does not succeed because it would require the conversion of \(f \ ?y\) and \(O\) where \(?y\) is a variable to instantiate. Tactic \texttt{simple apply} does not either traverse tuples as \texttt{apply} does.

Because it reasons modulo a limited amount of conversion, \texttt{simple apply} fails quicker than \texttt{apply} and it is then well-suited for uses in used-defined tactics that backtrack often.

6. \[simple\] \texttt{apply term\(_1\) \[with \ bindings\_list\(_1\)] , \ldots , term\(_n\) \[with \ bindings\_list\(_n\)]}

This summarizes the different syntaxes for \texttt{apply}.

7. \texttt{lapply term}

This tactic applies to any goal, say \(G\). The argument \(term\) has to be well-formed in the current context, its type being reducible to a non-dependent product \(A \rightarrow B\) with \(B\) possibly containing products. Then it generates two subgoals \(B \rightarrow G\) and \(A\). Applying \texttt{lapply H} (where \(H\) has type \(A \rightarrow B\) and \(B\) does not start with a product) does the same as giving the sequence \texttt{cut B.2:apply H.} where \texttt{cut} is described below.

\textbf{Warning:} When \(term\) contains more than one non dependent product the tactic \texttt{lapply} only takes into account the first product.

8.3.7 \texttt{set ( ident := term )}

This replaces \texttt{term} by \texttt{ident} in the conclusion or in the hypotheses of the current goal and adds the new definition \texttt{ident := term} to the local context. The default is to make this replacement only in the conclusion.

\textbf{Variants:}

1. \texttt{set ( ident := term ) in goal\_occurrences}

This notation allows to specify which occurrences of \(term\) have to be substituted in the context. The \texttt{in goal\_occurrences} clause is an occurrence clause whose syntax and behavior is described in Section 8.3.18.

2. \texttt{set ( ident binder \ldots binder := term )}

This is equivalent to \texttt{set ( ident := fun binder \ldots binder => term )}.

3. \texttt{set term}

This behaves as \texttt{set ( ident := term )} but \texttt{ident} is generated by COQ. This variant also supports an occurrence clause.
4. set (ident₀ binder ... binder := term) in goal_occurrences
set term in goal_occurrences

These are the general forms which combine the previous possibilities.

5. remember term as ident

This behaves as set (ident := term) in * and using a logical (Leibniz’s) equality instead of a local definition.

6. remember term as ident in goal_occurrences

This is a more general form of remember that remembers the occurrences of term specified by an occurrences set.

7. pose (ident := term)

This adds the local definition ident := term to the current context without performing any replacement in the goal or in the hypotheses. It is equivalent to set (ident := term) in |-.

8. pose (ident binder ... binder := term)

This is equivalent to pose (ident := fun binder ... binder => term).

9. pose term

This behaves as pose (ident := term) but ident is generated by Coq.

8.3.8 assert (ident : form)

This tactic applies to any goal. assert (H : U) adds a new hypothesis of name H asserting U to the current goal and opens a new subgoal U³. The subgoal U comes first in the list of subgoals remaining to prove.

Error messages:

1. Not a proposition or a type

Arises when the argument form is neither of type Prop, Set nor Type.

Variants:

1. assert form

This behaves as assert (ident : form) but ident is generated by Coq.

2. assert (ident := term)

This behaves as assert (ident : type); [exact term|idtac] where type is the type of term.

3. cut form

This tactic applies to any goal. It implements the non dependent case of the “App” rule given in Section 4.2. (This is Modus Ponens inference rule.) cut U transforms the current goal T into the two following subgoals: U -> T and U. The subgoal U -> T comes first in the list of remaining subgoal to prove.

³This corresponds to the cut rule of sequent calculus.
4. **assert** *form* by **tactic**

   This tactic behaves like `assert` but applies `tactic` to solve the subgoals generated by `assert`.

5. **assert** *form* as **intro_pattern**

   If `intro_pattern` is a naming introduction pattern (see Section 8.7.3), the hypothesis is named after this introduction pattern (in particular, if `intro_pattern` is `ident`, the tactic behaves like `assert (ident : form)`).

   If `intro_pattern` is a disjunctive/conjunctive introduction pattern, the tactic behaves like `assert form` then destructing the resulting hypothesis using the given introduction pattern.

6. **assert** *form* as **intro_pattern** by **tactic**

   This combines the two previous variants of `assert`.

7. **pose proof** *term* as **intro_pattern**

   This tactic behaves like `assert T as intro_pattern` by `exact term` where `T` is the type of `term`.

   In particular, `pose proof term as ident` behaves as `assert (ident:T)` by `exact term` (where `T` is the type of `term`) and `pose proof term as disj_conj_intro_pattern` behaves like `destruct term` as `disj_conj_intro_pattern`.

8. **specialize** (ident term_1 ... term_n)

   specialize ident with bindings_list

   The tactic `specialize` works on local hypothesis `ident`. The premises of this hypothesis (either universal quantifications or non-dependent implications) are instantiated by concrete terms coming either from arguments `term_1 ... term_n` or from a bindings list (see Section 8.3.17 for more about bindings lists). In the second form, all instantiation elements must be given, whereas in the first form the application to `term_1 ... term_n` can be partial. The first form is equivalent to `assert (ident' := ident term_1 ... term_n); clear ident; rename ident' into ident`.

   The name `ident` can also refer to a global lemma or hypothesis. In this case, for compatibility reasons, the behavior of `specialize` is close to that of `generalize`: the instantiated statement becomes an additional premise of the goal.

---

8.3.9 **apply** *term* in **ident**

This tactic applies to any goal. The argument `term` is a term well-formed in the local context and the argument `ident` is an hypothesis of the context. The tactic `apply term in ident` tries to match the conclusion of the type of `ident` against a non dependent premise of the type of `term`, trying them from right to left. If it succeeds, the statement of hypothesis `ident` is replaced by the conclusion of the type of `term`. The tactic also returns as many subgoals as the number of other non dependent premises in the type of `term` and of the non dependent premises of the type of `ident`. If the conclusion of the type of `term` does not match the goal and the conclusion is an inductive type isomorphic to a tuple type, then the tuple is (recursively) decomposed and the first component of the tuple of which a non dependent premise matches the conclusion of the type of `ident`. Tuples are decomposed in a width-first left-to-right order (for instance if the type of `H1` is a `A <-> B` statement, and the type of `H2` is `A` then `apply H1`
in $H2$ transforms the type of $H2$ into $B$). The tactic apply relies on first-order pattern-matching with dependent types.

**Error messages:**

1. **Statement without assumptions**
   
   This happens if the type of term has no non dependent premise.

2. **Unable to apply**

   This happens if the conclusion of ident does not match any of the non dependent premises of the type of term.

**Variants:**

1. **apply term , . . . , term in ident**

   This applies each of term in sequence in ident.

2. **apply term bindings_list , . . . , term bindings_list in ident**

   This does the same but uses the bindings in each bindings_list to instantiate the parameters of the corresponding type of term (see syntax of bindings in Section 8.3.17).

3. **eapply term bindings_list , . . . , term bindings_list in ident**

   This works as apply term bindings_list , . . . , term bindings_list in ident but turns unresolved bindings into existential variables, if any, instead of failing.

4. **apply term, bindings_list , . . . , term, bindings_list in ident as disj_conj_intro_pattern**

   This works as apply term, bindings_list , . . . , term, bindings_list in ident then destructs the hypothesis ident along disj_conj_intro_pattern as destruct ident as disj_conj_intro_pattern would.

5. **eapply term, bindings_list , . . . , term, bindings_list in ident as disj_conj_intro_pattern**

   This works as apply term, bindings_list , . . . , term, bindings_list in ident as disj_conj_intro_pattern but using eapply.

6. **simple apply term in ident**

   This behaves like apply term in ident but it reasons modulo conversion only on subterms that contain no variables to instantiate. For instance, if id := fun x:nat => x and H : forall y, id y = y -> True and H0 : O = O then simple apply H in H0 does not succeed because it would require the conversion of $f$ ?y and O where ?y is a variable to instantiate. Tactic simple apply term in ident does not either traverse tuples as apply term in ident does.

7. **simple apply term, bindings_list , . . . , term, bindings_list in ident as disj_conj_intro_pattern**

   This are the general forms of simple apply term in ident and simple eapply term in ident.


8.3.10 generalize term

This tactic applies to any goal. It generalizes the conclusion w.r.t. one subterm of it. For example:

Coq < Show.
1 subgoal
x : nat
y : nat
============================
0 <= x + y + y

Coq < generalize (x + y + y).
1 subgoal
x : nat
y : nat
============================
forall n : nat, 0 <= n

If the goal is $G$ and $t$ is a subterm of type $T$ in the goal, then generalize $t$ replaces the goal by $\forall n : T, G'$ where $G'$ is obtained from $G$ by replacing all occurrences of $t$ by $x$. The name of the variable (here $n$) is chosen based on $T$.

Variants:

1. generalize $term_1, \ldots, term_n$
   
   Is equivalent to generalize $term_n; \ldots; generalize term_1$. Note that the sequence of $term_i$'s are processed from $n$ to 1.

2. generalize $term$ at $num_1 \ldots num_i$
   
   Is equivalent to generalize $term$ but generalizing only over the specified occurrences of $term$ (counting from left to right on the expression printed using option Set Printing All).

3. generalize $term$ as $ident$
   
   Is equivalent to generalize $term$ but use $ident$ to name the generalized hypothesis.

4. generalize $term_1$ at $num_{i_1} \ldots num_{i_i}$ as $ident_1, \ldots, term_n$ at $num_{n_1} \ldots num_{n_i}$ as $ident_2$
   
   This is the most general form of generalize that combines the previous behaviors.

5. generalize dependent $term$
   
   This generalizes $term$ but also all hypotheses which depend on $term$. It clears the generalized hypotheses.

6. revert $ident_1 \ldots ident_n$
   
   This is equivalent to a generalize followed by a clear on the given hypotheses. This tactic can be seen as reciprocal to intros.
8.3.11 change \textit{term}

This tactic applies to any goal. It implements the rule “Conv” given in Section 4.3. change \( U \) replaces the current goal \( T \) with \( U \) providing that \( U \) is well-formed and that \( T \) and \( U \) are convertible.

\textbf{Error messages:}

1. Not convertible

\textbf{Variants:}

1. change \( \textit{term}_1 \) with \( \textit{term}_2 \)
   
   This replaces the occurrences of \( \textit{term}_1 \) by \( \textit{term}_2 \) in the current goal. The terms \( \textit{term}_1 \) and \( \textit{term}_2 \) must be convertible.

2. change \( \textit{term}_1 \) at \( \text{num}_1 \ldots \text{num}_i \) with \( \textit{term}_2 \)
   
   This replaces the occurrences numbered \( \text{num}_1 \ldots \text{num}_i \) of \( \textit{term}_1 \) by \( \textit{term}_2 \) in the current goal. The terms \( \textit{term}_1 \) and \( \textit{term}_2 \) must be convertible.

   \textbf{Error message:} Too few occurrences

3. change \( \textit{term} \) in \( \textit{ident} \)

4. change \( \textit{term}_1 \) with \( \textit{term}_2 \) in \( \textit{ident} \)

5. change \( \textit{term}_1 \) at \( \text{num}_1 \ldots \text{num}_i \) with \( \textit{term}_2 \) in \( \textit{ident} \)

   This applies the change tactic not to the goal but to the hypothesis \( \textit{ident} \).

\textbf{See also:} 8.5

8.3.12 fix \textit{ident} \textit{num}

This tactic is a primitive tactic to start a proof by induction. In general, it is easier to rely on higher-level induction tactics such as the ones described in Section 8.7.

In the syntax of the tactic, the identifier \( \textit{ident} \) is the name given to the induction hypothesis. The natural number \( \textit{num} \) tells on which premise of the current goal the induction acts, starting from 1 and counting both dependent and non dependent products. Especially, the current lemma must be composed of at least \( \textit{num} \) products.

Like in a fix expression, the induction hypotheses have to be used on structurally smaller arguments. The verification that inductive proof arguments are correct is done only at the time of registering the lemma in the environment. To know if the use of induction hypotheses is correct at some time of the interactive development of a proof, use the command \texttt{Guarded} (see Section 7.3.2).

\textbf{Variants:}

1. \texttt{fix \textit{ident}_1 \ textit{num} with ( \textit{ident}_2 \ \texttt{binder}_2 \ldots \texttt{binder}_2 \{ \text{struct \ \textit{ident}'_2 \} \} : \textit{type}_2 \} \ldots ( \textit{ident}_1 \ \texttt{binder}_n \ldots \texttt{binder}_n \{ \text{struct \ \textit{ident}'_n \} \} : \textit{type}_n \)}

   This starts a proof by mutual induction. The statements to be simultaneously proved are respectively \( \forall \texttt{binder}_2 \ldots \texttt{binder}_2, \ \textit{type}_2, \ldots, \forall \texttt{binder}_n \ldots \texttt{binder}_n, \ \textit{type}_n \). The identifiers \( \textit{ident}_1 \ldots \textit{ident}_n \) are the names of the induction hypotheses. The identifiers \( \textit{ident}'_2 \ldots \textit{ident}'_n \) are the respective names of the premises on which the induction is performed in the statements to be simultaneously proved (if not given, the system tries to guess itself what they are).
8.3 Basics

8.3.13 **cofix** *ident*

This tactic starts a proof by coinduction. The identifier *ident* is the name given to the coinduction hypothesis. Like in a *cofix* expression, the use of induction hypotheses have to guarded by a constructor. The verification that the use of coinductive hypotheses is correct is done only at the time of registering the lemma in the environment. To know if the use of coinduction hypotheses is correct at some time of the interactive development of a proof, use the command Guarded (see Section 7.3.2).

**Variants:**

1. **cofix** *ident*$_1$ with ( *ident*$_2$ *binder*$_2$ ... *binder*$_2$ : *type*$_2$ ) ... ( *ident*$_1$ *binder*$_1$ ... *binder*$_1$ : *type*$_n$ )

   This starts a proof by mutual coinduction. The statements to be simultaneously proved are respectively forall *binder*$_2$ ... *binder*$_2$, *type*$_2$, ..., forall *binder*$_n$ ... *binder*$_n$, *type*$_n$. The identifiers *ident*$_1$ ... *ident*$_n$ are the names of the coinduction hypotheses.

8.3.14 **evar** (*ident* : *term*)

The *evar* tactic creates a new local definition named *ident* with type *term* in the context. The body of this binding is a fresh existential variable.

8.3.15 **instantiate** (*num* := *term*)

The *instantiate* tactic allows to solve an existential variable with the term *term*. The *num* argument is the position of the existential variable from right to left in the conclusion. This cannot be the number of the existential variable since this number is different in every session.

**Variants:**

1. **instantiate** (*num* := *term*) in *ident*

2. **instantiate** (*num* := *term*) in (Value of *ident*)

3. **instantiate** (*num* := *term*) in (Type of *ident*)

   These allow to refer respectively to existential variables occurring in a hypothesis or in the body or the type of a local definition.

4. **instantiate**

   Without argument, the *instantiate* tactic tries to solve as many existential variables as possible, using information gathered from other tactics in the same tactical. This is automatically done after each complete tactic (i.e. after a dot in proof mode), but not, for example, between each tactic when they are sequenced by semicolons.

8.3.16 **admit**

The *admit* tactic “solves” the current subgoal by an axiom. This typically allows to temporarily skip a subgoal so as to progress further in the rest of the proof. To know if some proof still relies on unproved subgoals, one can use the command Print Assumptions (see Section 6.2.5). Admitted subgoals have names of the form *ident*_admitted possibly followed by a number.
Tactics that take a term as argument may also support a bindings list, so as to instantiate some parameters of the term by name or position. The general form of a term equipped with a bindings list is \texttt{term with bindings_list} where \texttt{bindings_list} may be of two different forms:

- In a bindings list of the form \( \langle \text{ref}_1 := \text{term}_1 \rangle \ldots \langle \text{ref}_n := \text{term}_n \rangle \), \texttt{ref} is either an \texttt{ident} or a \texttt{num}. The references are determined according to the type of \texttt{term}. If \texttt{ref}_i is an identifier, this identifier has to be bound in the type of \texttt{term} and the binding provides the tactic with an instance for the parameter of this name. If \texttt{ref}_i is some number \( n \), this number denotes the \( n \)-th non dependent premise of the \texttt{term}, as determined by the type of \texttt{term}.

\textbf{Error message:} No such binder

- A bindings list can also be a simple list of terms \texttt{term}_1 \ldots \texttt{term}_n. In that case the references to which these terms correspond are determined by the tactic. In case of \texttt{induction}, \texttt{destruct}, \texttt{elim} and \texttt{case} (see Section 11) the terms have to provide instances for all the dependent products in the type of \texttt{term} while in the case of \texttt{apply}, or of \texttt{constructor} and its variants, only instances for the dependent products which are not bound in the conclusion of the type are required.

\textbf{Error message:} Not the right number of missing arguments

### 8.3.18 Occurrences sets and occurrences clauses

An occurrences clause is a modifier to some tactics that obeys the following syntax:

\[
\text{occurrence_clause ::= in goal_occurrences}
\]

\[
\text{goal_occurrences ::= [ident}_1 [at_occurrences], \ldots, \ident_m [at_occurrences]] \mid \star \mid [\star [at_occurrences]]
\]

\[
\text{at_occurrences ::= at occurrences}
\]

\[
\text{occurrences ::= [at num}_1 \ldots \text{num}_n\]
\]

The role of an occurrence clause is to select a set of occurrences of a \texttt{term} in a goal. In the first case, the \texttt{ident}_i \texttt{[at num}_1 \ldots \texttt{num}_i\] parts indicate that occurrences have to be selected in the hypotheses named \texttt{ident}_i. If no numbers are given for hypothesis \texttt{ident}_i, then all occurrences of \texttt{term} in the hypothesis are selected. If numbers are given, they refer to occurrences of \texttt{term} when the term is printed using option \texttt{Set Printing All} (see Section 2.9), counting from left to right. In particular, occurrences of \texttt{term} in implicit arguments (see Section 2.7) or coercions (see Section 2.8) are counted.

If a minus sign is given between \texttt{at} and the list of occurrences, it negates the condition so that the clause denotes all the occurrences except the ones explicitly mentioned after the minus sign.

As an exception to the left-to-right order, the occurrences in the \texttt{return} subexpression of a \texttt{match} are considered before the occurrences in the matched term.

In the second case, the \texttt{*} on the left of \texttt{|} means that all occurrences of \texttt{term} are selected in every hypothesis.

In the first and second case, if \texttt{*} is mentioned on the right of \texttt{|}, the occurrences of the conclusion of the goal have to be selected. If some numbers are given, then only the occurrences denoted by these numbers are selected. In no numbers are given, all occurrences of \texttt{term} in the goal are selected.
Finally, the last notation is an abbreviation for $\ast \vdash \ast$. Note also that $\vdash$ is optional in the first case when no $\ast$ is given.

Here are some tactics that understand occurrences clauses: set, remember, induction, destruct.

See also: Sections 8.3.7, 8.7, 2.9.

8.4 Negation and contradiction

8.4.1 absurd term

This tactic applies to any goal. The argument term is any proposition $P$ of type $\text{Prop}$. This tactic applies False elimination, that is it deduces the current goal from False, and generates as subgoals $\neg P$ and $P$. It is very useful in proofs by cases, where some cases are impossible. In most cases, $P$ or $\neg P$ is one of the hypotheses of the local context.

8.4.2 contradiction

This tactic applies to any goal. The contradiction tactic attempts to find in the current context (after all intros) one hypothesis which is equivalent to False. It permits to prune irrelevant cases. This tactic is a macro for the tactics sequence intros; elimtype False; assumption.

Error messages:

1. No such assumption

Variants:

1. contradiction ident

   The proof of False is searched in the hypothesis named ident.

8.4.3 contradict ident

This tactic allows to manipulate negated hypothesis and goals. The name ident should correspond to a hypothesis. With contradict $H$, the current goal and context is transformed in the following way:

- $H: \neg A \vdash B$ becomes $\vdash A$
- $H: \neg A \vdash \neg B$ becomes $H: B \vdash A$
- $H: A \vdash B$ becomes $\vdash \neg A$
- $H: A \vdash \neg B$ becomes $H: B \vdash \neg A$

8.5 Conversion tactics

This set of tactics implements different specialized usages of the tactic change.

All conversion tactics (including change) can be parameterized by the parts of the goal where the conversion can occur. This is done using goal clauses which consists in a list of hypotheses and,
optionally, of a reference to the conclusion of the goal. For defined hypothesis it is possible to specify if the conversion should occur on the type part, the body part or both (default).

Goal clauses are written after a conversion tactic (tactics set 8.3.7, rewrite 8.8.1, replace 8.8.3 and autorewrite 8.12.12 also use goal clauses) and are introduced by the keyword in. If no goal clause is provided, the default is to perform the conversion only in the conclusion.

The syntax and description of the various goal clauses is the following:

\[\text{in } \text{ident}_1 \ldots \text{ident}_n |- \] only in hypotheses \text{ident}_1 \ldots \text{ident}_n

\[\text{in } \text{ident}_1 \ldots \text{ident}_n |- * \] in hypotheses \text{ident}_1 \ldots \text{ident}_n and in the conclusion

\[\text{in } * |- * \] in every hypothesis

\[\text{in } * (\text{equivalent to in } * |- *) \] everywhere

\[\text{in } (\text{type of } \text{ident}_1) (\text{value of } \text{ident}_2) \ldots |- \] in type part of \text{ident}_1, in the value part of \text{ident}_2, etc.

For backward compatibility, the notation \text{in } \text{ident}_1\ldots\text{ident}_n performs the conversion in hypotheses \text{ident}_1\ldots\text{ident}_n.

### 8.5.1 \text{cbv flag}_1 \ldots \text{flag}_n, \text{lazy flag}_1 \ldots \text{flag}_n and \text{compute}

These parameterized reduction tactics apply to any goal and perform the normalization of the goal according to the specified flags. In correspondence with the kinds of reduction considered in \text{COQ} namely $\beta$ (reduction of functional application), $\delta$ (unfolding of transparent constants, see 6.9.2), $\iota$ (contraction of local definitions), the flag are either beta, delta, iota or zeta. The delta flag itself can be refined into delta [qualid$_1$\ldotsqualid$_k$] or delta -[qualid$_1$\ldotsqualid$_k$], restricting in the first case the constants to unfold to the constants listed, and restricting in the second case the constant to unfold to all but the ones explicitly mentioned. Notice that the delta flag does not apply to variables bound by a let-in construction inside the term itself (use here the zeta flag). In any cases, opaque constants are not unfolded (see Section 6.9.1).

The goal may be normalized with two strategies: lazy (lazy tactic), or call-by-value (cbv tactic). The lazy strategy is a call-by-need strategy, with sharing of reductions: the arguments of a function call are partially evaluated only when necessary, and if an argument is used several times then it is computed only once. This reduction is efficient for reducing expressions with dead code. For instance, the proofs of a proposition exists $x. P(x)$ reduce to a pair of a witness $t$, and a proof that $t$ satisfies the predicate $P$. Most of the time, $t$ may be computed without computing the proof of $P(t)$, thanks to the lazy strategy.

The call-by-value strategy is the one used in ML languages: the arguments of a function call are evaluated first, using a weak reduction (no reduction under the $\lambda$-abstractions). Despite the lazy strategy always performs fewer reductions than the call-by-value strategy, the latter is generally more efficient for evaluating purely computational expressions (i.e. with few dead code).

**Variants:**

1. \text{compute}

\text{cbv}

These are synonyms for \text{cbv beta delta iota zeta}.
2. lazy
   This is a synonym for lazy beta delta iota zeta.

3. compute \([\text{qualid}_1 \ldots \text{qualid}_k]\)
   \(\text{cbv} \; [\text{qualid}_1 \ldots \text{qualid}_k]\)
   These are synonyms of \(\text{cbv} \; \text{beta} \; \text{delta} \; [\text{qualid}_1 \ldots \text{qualid}_k] \; \text{iota} \; \text{zeta}\).

4. compute \(-[\text{qualid}_1 \ldots \text{qualid}_k]\)
   \(\text{cbv} \; -[\text{qualid}_1 \ldots \text{qualid}_k]\)
   These are synonyms of \(\text{cbv} \; \text{beta} \; \text{delta} \; -[\text{qualid}_1 \ldots \text{qualid}_k] \; \text{iota} \; \text{zeta}\).

5. lazy \([\text{qualid}_1 \ldots \text{qualid}_k]\)
   lazy \(-[\text{qualid}_1 \ldots \text{qualid}_k]\)
   These are respectively synonyms of \(\text{cbv} \; \text{beta} \; \text{delta} \; [\text{qualid}_1 \ldots \text{qualid}_k] \; \text{iota} \; \text{zeta}\) and \(\text{cbv} \; \text{beta} \; \text{delta} \; -[\text{qualid}_1 \ldots \text{qualid}_k] \; \text{iota} \; \text{zeta}\).

6. \(\text{vm\_compute}\)
   This tactic evaluates the goal using the optimized call-by-value evaluation bytecode-based virtual machine. This algorithm is dramatically more efficient than the algorithm used for the \(\text{cbv}\) tactic, but it cannot be fine-tuned. It is specially interesting for full evaluation of algebraic objects. This includes the case of reflexion-based tactics.

8.5.2 red
This tactic applies to a goal which has the form \(\forall (x:T_1) \ldots (x_k:T_k), \; c \; t_1 \ldots \; t_n\) where \(c\) is a constant. If \(c\) is transparent then it replaces \(c\) with its definition (say \(t\)) and then reduces \((t \; t_1 \ldots \; t_n)\) according to \(\beta\nu\zeta\)-reduction rules.

Error messages:
1. Not reducible

8.5.3 hnf
This tactic applies to any goal. It replaces the current goal with its head normal form according to the \(\beta\nu\zeta\)-reduction rules, i.e. it reduces the head of the goal until it becomes a product or an irreducible term.

Example: The term \(\forall n : \text{nat}, \; (\text{plus} \; (S \; n) \; (S \; n))\) is not reduced by \(\text{hnf}\).

Remark: The \(\delta\) rule only applies to transparent constants (see Section 6.9.1 on transparency and opacity).

8.5.4 simpl
This tactic applies to any goal. The tactic \(\text{simpl}\) first applies \(\beta\iota\)-reduction rule. Then it expands transparent constants and tries to reduce \(T'\) according, once more, to \(\beta\iota\) rules. But when the \(\iota\) rule is not applicable then possible \(\delta\)-reductions are not applied. For instance trying to use \(\text{simpl}\) on \((\text{plus} \; n \; 0) = n\) changes nothing. Notice that only transparent constants whose name can be reused as such in the recursive calls are possibly unfolded. For instance a constant defined by \(\text{plus}' := \text{plus}\) is possibly...
unfolded and reused in the recursive calls, but a constant such as \( \text{succ} := \text{plus} (\text{S} \ 0) \) is never unfolded.

**Variants:**

1. **simpl \ term**
   
   This applies **simpl** only to the occurrences of \( \text{term} \) in the current goal.

2. **simpl \ term \ at \ num_1 \ldots \ num_i**
   
   This applies **simpl** only to the \( num_1, \ldots, num_i \) occurrences of \( \text{term} \) in the current goal.

   **Error message:** Too few occurrences

3. **simpl \ ident**
   
   This applies **simpl** only to the applicative subterms whose head occurrence is \( \text{ident} \).

4. **simpl \ ident \ at \ num_1 \ldots \ num_i**
   
   This applies **simpl** only to the \( num_1, \ldots, num_i \) applicative subterms whose head occurrence is \( \text{ident} \).

**8.5.5 unfold \ qualid**

This tactic applies to any goal. The argument \( \text{qualid} \) must denote a defined transparent constant or local definition (see Sections 1.3.2 and 6.9.2). The tactic **unfold** applies the \( \delta \) rule to each occurrence of the constant to which \( \text{qualid} \) refers in the current goal and then replaces it with its \( \beta\iota \)-normal form.

**Error messages:**

1. **\qualid \ does not denote an evaluable constant**

**Variants:**

1. **unfold \ qualid_1, \ldots, qualid_n**
   
   Replaces simultaneously \( \text{qualid}_1, \ldots, \text{qualid}_n \) with their definitions and replaces the current goal with its \( \beta\iota \) normal form.

2. **unfold \ qualid_1 \ at \ num_1^1, \ldots, num_i^1, \ldots, qualid_n \ at \ num_1^n \ldots num_j^n**
   
   The lists \( num_1^1, \ldots, num_i^1 \) and \( num_1^n, \ldots, num_j^n \) specify the occurrences of \( \text{qualid}_1, \ldots, \text{qualid}_n \) to be unfolded. Occurrences are located from left to right.

   **Error message:** bad occurrence number of \( \text{qualid}_i \)

   **Error message:** \( \text{qualid}_i \) does not occur

3. **unfold \ string**
   
   If \( \text{string} \) denotes the discriminating symbol of a notation (e.g. "+") or an expression defining a notation (e.g. "\_ + \_"), and this notation refers to an unfoldable constant, then the tactic unfolds it.
4. unfold string%key

   This is variant of unfold string where string gets its interpretation from the scope bound to the delimiting key key instead of its default interpretation (see Section 12.2.2).

5. unfold qualid_or_string\,\_at\,\, num_1, \ldots, num_i, \ldots, qualid_or_string\,\_\, at\,\, num_1^{\ldots \, \, num_j^\prime}

   This is the most general form, where qualid_or_string is either a qualid or a string referring to a notation.

8.5.6 fold term

   This tactic applies to any goal. The term term is reduced using the red tactic. Every occurrence of the resulting term in the goal is then replaced by term.

   Variants:
   1. fold term_1 ... term_n
      Equivalent to fold term_1; ...; fold term_n.

8.5.7 pattern term

   This command applies to any goal. The argument term must be a free subterm of the current goal. The command pattern performs β-expansion (the inverse of β-reduction) of the current goal (say T) by

   1. replacing all occurrences of term in T with a fresh variable
   2. abstracting this variable
   3. applying the abstracted goal to term

   For instance, if the current goal T is expressible has \( \phi(t) \) where the notation captures all the instances of t in \( \phi(t) \), then pattern t transforms it into (fun x:A :⇒ \( \phi(x) \)) t. This command can be used, for instance, when the tactic apply fails on matching.

   Variants:
   1. pattern term\,\at\,\, num_1 \ldots num_n
      Only the occurrences num_1 \ldots num_n of term are considered for β-expansion. Occurrences are located from left to right.
   2. pattern term\,\at\,\, -\,\, num_1 \ldots num_n
      All occurrences except the occurrences of indexes num_1 \ldots num_n of term are considered for β-expansion. Occurrences are located from left to right.
   3. pattern term_1, \ldots, term_m
      Starting from a goal \( \phi(t_1 \ldots t_m) \), the tactic pattern\,\, t_1, \ldots, \, t_m generates the equivalent goal (fun (x_1:A_1) \ldots (x_m:A_m) :⇒ \( \phi(x_1 \ldots x_m) \))\,\, t_1 \ldots \, t_m.
      If \( t_i \) occurs in one of the generated types A_j these occurrences will also be considered and possibly abstracted.
4. pattern term₁ at num₁₁ . . . num₁₁n₁, . . . , termₘ at numₘ₁ . . . numₘₙₘ

This behaves as above but processing only the occurrences num₁₁, . . . , num₁₁n₁ of term₁, . . . , numₘ, . . . , numₘₙₘ of termₘ starting from termₘ.

5. pattern term₁ [at [-] num₁₁ . . . num₁₁n₁], . . . , termₘ [at [-] numₘ₁ . . . numₘₙₘ]

This is the most general syntax that combines the different variants.

8.5.8 Conversion tactics applied to hypotheses

\texttt{conv\_tactic in ident₁ . . . identₙ}

Applies the conversion tactic \texttt{conv\_tactic} to the hypotheses \texttt{ident₁, . . . , identₙ}. The tactic \texttt{conv\_tactic} is any of the conversion tactics listed in this section.

If \texttt{identᵢ} is a local definition, then \texttt{identᵢ} can be replaced by (\texttt{Type of identᵢ}) to address not the body but the type of the local definition. Example: unfold not in (\texttt{Type of H₁}) (\texttt{Type of H₃}).

Error messages:

1. No such hypothesis: \texttt{identᵢ}.

8.6 Introductions

Introduction tactics address goals which are inductive constants. They are used when one guesses that the goal can be obtained with one of its constructors’ type.

8.6.1 constructor \texttt{num}

This tactic applies to a goal such that the head of its conclusion is an inductive constant (say \texttt{I}). The argument \texttt{num} must be less or equal to the numbers of constructor(s) of \texttt{I}. Let \texttt{ci} be the \texttt{i}-th constructor of \texttt{I}, then 
\texttt{constructor i is equivalent to intros; apply ci}.

Error messages:

1. Not an inductive product
2. Not enough constructors

Variants:

1. constructor

   This tries \texttt{constructor 1 then constructor 2, . . . , then constructor n} where \texttt{n} if the number of constructors of the head of the goal.

2. constructor \texttt{num with bindings\_list}

   Let \texttt{ci} be the \texttt{i}-th constructor of \texttt{I}, then 
\texttt{constructor i with bindings\_list is equivalent to intros; apply ci with bindings\_list}.

   \textbf{Warning}: the terms in the \texttt{bindings\_list} are checked in the context where \texttt{constructor} is executed and not in the context where \texttt{apply} is executed (the introductions are not taken into account).
3. split
Applies if I has only one constructor, typically in the case of conjunction \( A \land B \). Then, it is equivalent to constructor 1.

4. exists bindings_list
Applies if I has only one constructor, for instance in the case of existential quantification \( \exists x. P(x) \). Then, it is equivalent to intros; constructor 1 with bindings_list.

5. left
right
Apply if I has two constructors, for instance in the case of disjunction \( A \lor B \). Then, they are respectively equivalent to constructor 1 and constructor 2.

6. left bindings_list
right bindings_list
split bindings_list
As soon as the inductive type has the right number of constructors, these expressions are equivalent to the corresponding constructor i with bindings_list.

7. econstructor
eexists
esplit
eleft
eright

These tactics and their variants behave like constructor, exists, split, left, right and their variants but they introduce existential variables instead of failing when the instantiation of a variable cannot be found (cf eapply and Section 10.2).

8.7 Induction and Case Analysis

The tactics presented in this section implement induction or case analysis on inductive or coinductive objects (see Section 4.5).

8.7.1 induction term

This tactic applies to any goal. The type of the argument term must be an inductive constant. Then, the tactic induction generates subgoals, one for each possible form of term, i.e. one for each constructor of the inductive type.

The tactic induction automatically replaces every occurrences of term in the conclusion and the hypotheses of the goal. It automatically adds induction hypotheses (using names of the form IHn1) to the local context. If some hypothesis must not be taken into account in the induction hypothesis, then it needs to be removed first (you can also use the tactics elim or simple induction, see below).

There are particular cases:

- If term is an identifier ident denoting a quantified variable of the conclusion of the goal, then induction ident behaves as intros until ident; induction ident.
- If \textit{term} is a \textit{num}, then \texttt{induction num} behaves as \texttt{intros} until \textit{num} followed by \texttt{induction} applied to the last introduced hypothesis.

\textbf{Remark:} For simple induction on a numeral, use syntax \texttt{induction (num)} (not very interesting anyway).

\textbf{Example:}

Coq < Lemma induction_test : forall n:nat, n = n -> n <= n.
1 subgoal

\begin{verbatim}
============================
forall n : nat, n = n -> n <= n
\end{verbatim}

Coq < intros n H.
1 subgoal

\begin{verbatim}
n : nat
H : n = n
============================
n <= n
\end{verbatim}

Coq < induction n.
2 subgoals

\begin{verbatim}
H : 0 = 0
============================
0 <= 0
\end{verbatim}

\texttt{subgoal 2 is:}

\begin{verbatim}
S n <= S n
\end{verbatim}

\textbf{Error messages:}

1. Not an inductive product

2. Unable to find an instance for the variables \texttt{ident} \ldots \texttt{ident}

Use in this case the variant \texttt{elim} \ldots with \ldots below.

\textbf{Variants:}

1. \texttt{induction term as disj_conj_intro_pattern}

This behaves as \texttt{induction term} but uses the names in \texttt{disj_conj_intro_pattern} to name the variables introduced in the context. The \texttt{disj_conj_intro_pattern} must typically be of the form \{ \texttt{p11} \ldots \texttt{p1m1} | \ldots | \texttt{pm1} \ldots \texttt{pmmn} \} with \textit{m} being the number of constructors of the type of \textit{term}. Each variable introduced by \texttt{induction} in the context of the \textit{i}th goal gets its name from the list \texttt{p1} \ldots \texttt{pm}, in order. If there are not enough names, \texttt{induction} invents names for the remaining variables to introduce. More generally, the \texttt{p_{ij}} can be any disjunctive/conjunctive introduction pattern (see Section 8.7.3). For instance, for an inductive type with one constructor, the pattern notation \{ \texttt{p1}, \ldots, \texttt{pn} \} can be used instead of \{ \texttt{p1 \ldots pn} \}.
2. induction term as naming_intro_pattern
   This behaves as induction term but adds an equation between term and the value that term
takes in each of the induction case. The name of the equation is built according to naming_intro_pattern
which can be an identifier, a “?” etc, as indicated in Section 8.7.3.

3. induction term as naming_intro_pattern disj_conj_intro_pattern
   This combines the two previous forms.

4. induction term with bindings_list
   This behaves like induction term providing explicit instances for the premises of the type of
   term (see the syntax of bindings in Section 8.3.17).

5. einduction term
   This tactic behaves like induction term excepts that it does not fail if some dependent premise
   of the type of term is not inferable. Instead, the unresolved premises are posed as existential
   variables to be inferred later, in the same way as eapply does (see Section 10.2).

6. induction term₁ using term₂
   This behaves as induction term₁ but using term₂ as induction scheme. It does not expect the
   conclusion of the type of term₁ to be inductive.

7. induction term₁ using term₂ with bindings_list
   This behaves as induction term₁ using term₂ but also providing instances for the
   premises of the type of term₂.

8. induction term₁ ... termₙ using qualid
   This syntax is used for the case qualid denotes an induction principle with complex predicates as
   the induction principles generated by Function or Functional Scheme may be.

9. induction term in goal_occurrences
   This syntax is used for selecting which occurrences of term the induction has to be carried on.
   The in at_occurrences clause is an occurrence clause whose syntax and behavior is described in
   Section 8.3.18.
   When an occurrence clause is given, an equation between term and the value it gets in each case
   of the induction is added to the context of the subgoals corresponding to the induction cases (even
   if no clause as naming_intro_pattern is given).

10. induction term₁ with bindings_list₁ as naming_intro_pattern disj_conj_intro_pattern
    using term₂ with bindings_list₂ in goal_occurrences
    einduction term₁ with bindings_list₁ as naming_intro_pattern
    disj_conj_intro_pattern using term₂ with bindings_list₂ in goal_occurrences
    These are the most general forms of induction and einduction. It combines the effects of
    the with, as, using, and in clauses.

11. elim term
    This is a more basic induction tactic. Again, the type of the argument term must be an inductive
type. Then, according to the type of the goal, the tactic elim chooses the appropriate destructor
and applies it as the tactic apply would do. For instance, if the proof context contains \texttt{n:nat}
and the current goal is \texttt{T} of type \texttt{Prop}, then \texttt{elim n} is equivalent to \texttt{apply nat_ind with (n:=n)}. The tactic \texttt{elim} does not modify the context of the goal, neither introduces the induction
loading into the context of hypotheses.

More generally, \texttt{elim term} also works when the type of \texttt{term} is a statement with premises and
whose conclusion is inductive. In that case the tactic performs induction on the conclusion of
the type of \texttt{term} and leaves the non-dependent premises of the type as subgoals. In the case of
dependent products, the tactic tries to find an instance for which the elimination lemma applies
and fails otherwise.

12. \texttt{elim term with bindings_list}

Allows to give explicit instances to the premises of the type of \texttt{term} (see Section 8.3.17).

13. \texttt{eelim term}

In case the type of \texttt{term} has dependent premises, this turns them into existential variables to be
resolved later on.

14. \texttt{elim term_1 using term_2}
\texttt{elim term_1 using term_2 with bindings_list}

Allows the user to give explicitly an elimination predicate \texttt{term_2} which is not the standard one for
the underlying inductive type of \texttt{term_1}. The \texttt{bindings_list} clause allows to instantiate premises of
the type of \texttt{term_2}.

15. \texttt{elim term_1 with bindings_list_1 using term_2 with bindings_list_2}
\texttt{eelim term_1 with bindings_list_1 using term_2 with bindings_list_2}

These are the most general forms of \texttt{elim} and \texttt{eelim}. It combines the effects of the using
clause and of the two uses of the with clause.

16. \texttt{elimtype form}

The argument \texttt{form} must be inductively defined. \texttt{elimtype I} is equivalent to \texttt{cut I \ intro Hn; elim Hn; clear Hn}. Therefore the hypothesis \texttt{Hn} will not appear in the context(s) of
the subgoal(s). Conversely, if \texttt{t} is a term of (inductive) type \texttt{I} and which does not occur in the
goal then \texttt{elim t} is equivalent to \texttt{elimtype I; 2: exact t}.

17. \texttt{simple induction ident}

This tactic behaves as \texttt{intros until ident; elim ident} when \texttt{ident} is a quantified variable
of the goal.

18. \texttt{simple induction num}

This tactic behaves as \texttt{intros until num; elim ident} where \texttt{ident} is the name given by
\texttt{intros until num} to the \texttt{num}-th non-dependent premise of the goal.

8.7.2 \texttt{destruct term}

The tactic \texttt{destruct} is used to perform case analysis without recursion. Its behavior is similar to
\texttt{induction} except that no induction hypothesis is generated. It applies to any goal and the type of
\texttt{term} must be inductively defined. There are particular cases:
8.7 Induction and Case Analysis

- If \texttt{term} is an identifier \texttt{ident} denoting a quantified variable of the conclusion of the goal, then \texttt{destruct ident} behaves as \texttt{intros until ident; destruct ident}.

- If \texttt{term} is a \texttt{num}, then \texttt{destruct num} behaves as \texttt{intros until num} followed by \texttt{destruct} applied to the last introduced hypothesis.

\textbf{Remark}: For destruction of a numeral, use syntax \texttt{destruct (num)} (not very interesting anyway).

\textbf{Variants}:

1. \texttt{destruct term as disj_conj_intro_pattern}
   This behaves as \texttt{destruct term} but uses the names in \texttt{intro_pattern} to name the variables introduced in the context. The \texttt{intro_pattern} must have the form \([p_1 \ldots p_{i_1} | \ldots | p_m \ldots p_{m_m}]\) with \(m\) being the number of constructors of the type of \texttt{term}. Each variable introduced by \texttt{destruct} in the context of the \(i^{th}\) goal gets its name from the list \(p_{i_1} \ldots p_{i_n}\) in order. If there are not enough names, \texttt{destruct} invents names for the remaining variables to introduce. More generally, the \(p_{ij}\) can be any disjunctive/conjunctive introduction pattern (see Section 8.7.3). This provides a concise notation for nested destruction.

2. \texttt{destruct term as disj_conj_intro_pattern_eqn}
   This behaves as \texttt{destruct term} but adds an equation between \texttt{term} and the value that \texttt{term} takes in each of the possible cases. The name of the equation is chosen by Coq. If \texttt{disj_conj_intro_pattern} is simply \([\ ]\), it is automatically considered as a disjunctive pattern of the appropriate size.

3. \texttt{destruct term as disj_conj_intro_pattern_eqn: naming_intro_pattern}
   This behaves as \texttt{destruct term as disj_conj_intro_pattern_eqn} but use \texttt{naming_intro_pattern} to name the equation (see Section 8.7.3). Note that spaces can generally be removed around \_eqn.

4. \texttt{destruct term with bindings_list}
   This behaves like \texttt{destruct term} providing explicit instances for the dependent premises of the type of \texttt{term} (see syntax of bindings in Section 8.3.17).

5. \texttt{edestruct term}
   This tactic behaves like \texttt{destruct term} excepts that it does not fail if the instance of a dependent premises of the type of \texttt{term} is not inferable. Instead, the unresolved instances are left as existential variables to be inferred later, in the same way as \texttt{eapply} does (see Section 10.2).

6. \texttt{destruct term\_1 using term\_2}
   \texttt{destruct term\_1 using term\_2 with bindings_list}
   These are synonyms of \texttt{induction term\_1 using term\_2} and \texttt{induction term\_1 using term\_2 with bindings_list}.

7. \texttt{destruct term in goal\_occurrences}
This syntax is used for selecting which occurrences of term the case analysis has to be done on. The in goal_occurrences clause is an occurrence clause whose syntax and behavior is described in Section 8.3.18.

When an occurrence clause is given, an equation between term and the value it gets in each case of the analysis is added to the context of the subgoals corresponding to the cases (even if no clause as naming_intro_pattern is given).

8. destruct term_1 with bindings_list_1 as disj_conj_intro_pattern _eqn: naming_intro_pattern using term_2 with bindings_list_2 in goal_occurrences
edestruct term_1 with bindings_list_1 as disj_conj_intro_pattern _eqn: naming_intro_pattern using term_2 with bindings_list_2 in goal_occurrences

These are the general forms of destruct and edestruct. They combine the effects of the with, as, using, and in clauses.

9. case term
The tactic case is a more basic tactic to perform case analysis without recursion. It behaves as elim term but using a case-analysis elimination principle and not a recursive one.

10. case_eq term
The tactic case_eq is a variant of the case tactic that allow to perform case analysis on a term without completely forgetting its original form. This is done by generating equalities between the original form of the term and the outcomes of the case analysis. The effect of this tactic is similar to the effect of destruct term in |- * with the exception that no new hypotheses are introduced in the context.

11. case term with bindings_list
Analogous to elim term with bindings_list above.

12. ecase term
ecase term with bindings_list
In case the type of term has dependent premises, or dependent premises whose values are not inferable from the with bindings_list clause, ecase turns them into existential variables to be resolved later on.

13. simple destruct ident
This tactic behaves as intros until ident; case ident when ident is a quantified variable of the goal.

14. simple destruct num
This tactic behaves as intros until num; case ident where ident is the name given by intros until num to the num-th non-dependent premise of the goal.

8.7.3 intros intro_pattern ... intro_pattern
This extension of the tactic intros combines introduction of variables or hypotheses and case analysis. An introduction pattern is either:

- A naming introduction pattern, i.e. either one of:
8.7 Induction and Case Analysis

- the pattern `?`
- the pattern `?ident`
- an identifier

- **A disjunctive/conjunctive introduction pattern**, i.e. either one of:
  - a disjunction of lists of patterns: `[p_{11} \ldots p_{1m_1} | \ldots | p_{11} \ldots p_{nm_n}]`
  - a conjunction of patterns: `\( (p_1, \ldots, p_n) \)`
  - a list of patterns `\( (p_1 \& \ldots \& p_n) \)` for sequence of right-associative binary constructs

- the wildcard: `_`
- the rewriting orientations: `->` or `<-`

Assuming a goal of type `Q \rightarrow P` (non dependent product), or of type `\forall x : T, P` (dependent product), the behavior of `intros p` is defined inductively over the structure of the introduction pattern `p`:

- introduction on `?` performs the introduction, and lets Coq choose a fresh name for the variable;
- introduction on `?ident` performs the introduction, and lets Coq choose a fresh name for the variable based on `ident`;
- introduction on `ident` behaves as described in Section 8.3.5;
- introduction over a disjunction of list of patterns \([p_{11} \ldots p_{1m_1} | \ldots | p_{11} \ldots p_{nm_n}]\) expects the product to be over an inductive type whose number of constructors is `n` (or more generally over a type of conclusion an inductive type built from `n` constructors, e.g. `C \rightarrow A \| B` if `n = 2`); it destructs the introduced hypothesis as `destruct` (see Section 8.7.2) would and applies on each generated subgoal the corresponding tactic; `intros p_{11} \ldots p_{nm_n}`; if the disjunctive pattern is part of a sequence of patterns and is not the last pattern of the sequence, then Coq completes the pattern so as all the argument of the constructors of the inductive type are introduced (for instance, the list of patterns `[ | ] H` applied on goal `\forall x : \text{nat}, x=0 \rightarrow 0=x` behaves the same as the list of patterns `[ | ? ] H`);
- introduction over a conjunction of patterns \((p_{11}, \ldots, p_{n})\) expects the goal to be a product over an inductive type `I` with a single constructor that itself has at least `n` arguments: it performs a case analysis over the hypothesis, as `destruct` would, and applies the patterns `p_{11} \ldots p_{n}` to the arguments of the constructor of `I` (observe that \((p_{11}, \ldots, p_{n})\) is an alternative notation for \([p_{11} \ldots p_{n}]\));
- introduction via \((p_1 \& \ldots \& p_n)\) is a shortcut for introduction via \((p_1, (\ldots, (\ldots, p_n)\ldots))\); it expects the hypothesis to be a sequence of right-associative binary inductive constructors such as `conj` or `ex_intro`; for instance, an hypothesis with type `A \& \exists x : B \& C \& D` can be introduced via pattern `\((a \& x \& b \& c \& d)\)`;
- introduction on the wildcard depends on whether the product is dependent or not: in the non dependent case, it erases the corresponding hypothesis (i.e. it behaves as an `intro` followed by a `clear`, cf Section 8.3.2) while in the dependent case, it succeeds and erases the variable only if the wildcard is part of a more complex list of introduction patterns that also erases the hypotheses depending on this variable;
• introduction over \( \rightarrow \) (respectively \( \leftarrow \)) expects the hypothesis to be an equality and the right-hand-side (respectively the left-hand-side) is replaced by the left-hand-side (respectively the right-hand-side) in both the conclusion and the context of the goal; if moreover the term to substitute is a variable, the hypothesis is removed.

**Remark:** \texttt{intros \( p_1 \ldots p_n \)} is not equivalent to \texttt{intros \( p_1; \ldots; intros \ p_n \)} for the following reasons:

• A wildcard pattern never succeeds when applied isolated on a dependent product, while it succeeds as part of a list of introduction patterns if the hypotheses that depends on it are erased too.

• A disjunctive or conjunctive pattern followed by an introduction pattern forces the introduction in the context of all arguments of the constructors before applying the next pattern while a terminal disjunctive or conjunctive pattern does not. Here is an example

```
Coq < Goal forall n:nat, n = 0 -> n = 0.
1 subgoal

    ============================
    forall n : nat, n = 0 -> n = 0
Coq < intros [ | ] H.
2 subgoals

    H : 0 = 0
    ============================
    0 = 0
subgoal 2 is:
  S n = 0
Coq < Show 2.
subgoal 2 is:

    n : nat
    H : S n = 0
    ============================
    S n = 0
Coq < Undo.
1 subgoal

    ============================
    forall n : nat, n = 0 -> n = 0
Coq < intros [ | ]; intros H.
2 subgoals

    H : 0 = 0
    ============================
    0 = 0
subgoal 2 is:
  S H = 0 -> S H = 0
Coq < Show 2.
subgoal 2 is:
```

Coq Reference Manual, V8.2pl2, June 29, 2010
\[ H : \text{nat} \]
\[ \text{============================} \]
\[ S \; H = 0 \rightarrow S \; H = 0 \]

Coq < Lemma intros_test : forall A B C:Prop, A \lor B \land C \rightarrow (A \rightarrow C) \rightarrow C.
1 subgoal

\[ \text{============================} \]
\[ \forall A \; B \; C : \text{Prop}, A \lor B \land C \rightarrow (A \rightarrow C) \rightarrow C \]
Coq < intros A B C [a| [_ c]] f.
2 subgoals

\[ \begin{align*}
A & : \text{Prop} \\
B & : \text{Prop} \\
C & : \text{Prop} \\
a & : A \\
f & : A \rightarrow C \\
\end{align*} \]
\[ \text{============================} \]
\[ C \]

subgoal 2 is:
\[ C \]

Coq < apply (f a).
1 subgoal

\[ \begin{align*}
A & : \text{Prop} \\
B & : \text{Prop} \\
C & : \text{Prop} \\
c & : C \\
f & : A \rightarrow C \\
\end{align*} \]
\[ \text{============================} \]
\[ C \]

Coq < exact c.
Proof completed.

Coq < Qed.
intros A B C [a| [_, c]] f.
apply (f a).

exact c.

intros_test is defined

8.7.4 double induction \(\text{ident}_1, \text{ident}_2\)

This tactic applies to any goal. If the variables \(\text{ident}_1\) and \(\text{ident}_2\) of the goal have an inductive type, then this tactic performs double induction on these variables. For instance, if the current goal is
\[ \forall n \; m : \text{nat}, \; P \; n \; m \rightarrow \text{double induction} \; n \; m \text{ yields the four cases with their respective inductive hypotheses.} \]

In particular, for proving \((P \; (S \; n) \; (S \; m))\), the generated induction hypotheses are \((P \; (S \; n) \; m)\) and \((m : \text{nat}) \; (P \; n \; m)\) (of the latter, \((P \; n \; m)\) and \((P \; n \; (S \; m))\) are derivable).
**Remark:** When the induction hypothesis \((P \ (S \ n) \ m)\) is not needed, induction \(\text{ident}_1\); destruct \(\text{ident}_2\) produces more concise subgoals.

**Variant:**

1. double induction \(\text{num}_1 \ \text{num}_2\)

   This applies double induction on the \(\text{num}_1^{th}\) and \(\text{num}_2^{th}\) non dependent premises of the goal. More generally, any combination of an \(\text{ident}\) and a \(\text{num}\) is valid.

### 8.7.5 dependent induction \(\text{ident}\)

The experimental tactic dependent induction performs induction-inversion on an instantiated inductive predicate. One needs to first require the Coq.Program.Equality module to use this tactic. The tactic is based on the BasicElim tactic by Conor McBride [97] and the work of Cristina Cornes around inversion [35]. From an instantiated inductive predicate and a goal it generates an equivalent goal where the hypothesis has been generalized over its indexes which are then constrained by equalities to be the right instances. This permits to state lemmas without resorting to manually adding these equalities and still get enough information in the proofs. A simple example is the following:

```coq
Coq < Lemma le_minus : forall n:nat, n < 1 -> n = 0.
1 subgoal

============================
forall n : nat, n < 1 -> n = 0
Coq < intros n H ; induction H.
2 subgoals

n : nat

n = 0
subgoal 2 is:
  n = 0

Here we didn’t get any information on the indexes to help fulfill this proof. The problem is that when we use the induction tactic we lose information on the hypothesis instance, notably that the second argument is 1 here. Dependent induction solves this problem by adding the corresponding equality to the context.

Coq < Lemma le_minus : forall n:nat, n < 1 -> n = 0.
1 subgoal

============================
forall n : nat, n < 1 -> n = 0
Coq < intros n H ; dependent induction H.
2 subgoals

0 = 0
subgoal 2 is:
  n = 0
```

"
The subgoal is cleaned up as the tactic tries to automatically simplify the subgoals with respect to the generated equalities. In this enriched context it becomes possible to solve this subgoal.

Coq < reflexivity.
1 subgoal

n : nat
H : S n <= 0
IHle : 0 = 1 -> n = 0

============================
n = 0

Now we are in a contradictory context and the proof can be solved.

Coq < inversion H.
Proof completed.

This technique works with any inductive predicate. In fact, the dependent induction tactic is just a wrapper around the induction tactic. One can make its own variant by just writing a new tactic based on the definition found in Coq.Program.Equality. Common useful variants are the following, defined in the same file:

**Variants:**

1. dependent induction ident generalizing ident\_1 \ldots ident\_n

   Does dependent induction on the hypothesis ident but first generalizes the goal by the given variables so that they are universally quantified in the goal. This is generally what one wants to do with the variables that are inside some constructors in the induction hypothesis. The other ones need not be further generalized.

2. dependent destruction ident

   Does the generalization of the instance ident but uses destruct instead of induction on the generalized hypothesis. This gives results equivalent to inversion or dependent inversion if the hypothesis is dependent.

A larger example of dependent induction and an explanation of the underlying technique are developed in section 10.6.

**8.7.6 decompose [ qualid\_1 \ldots qualid\_n ] term**

This tactic allows to recursively decompose a complex proposition in order to obtain atomic ones. Example:

Coq < Lemma ex1 : forall A B C:Prop, A \land B \land C \lor B \land C \lor C \land A -> C.
1 subgoal

============================

forall A B C : Prop, A \land B \land C \lor B \land C \lor C \land A -> C

Coq < intros A B C H; decompose [and or] H; assumption.
Proof completed.
Coq < Qed.
decompose does not work on right-hand sides of implications or products.

**Variants:**

1. **decompose sum** `term` This decomposes sum types (like `or`).

2. **decompose record** `term` This decomposes record types (inductive types with one constructor, like `and`, `and` and `exists` and those defined with the `Record` macro, see Section 2.1).

### 8.7.7 functional induction `(qualid term_1 \ldots term_n)`.

The *experimental* tactic `functional induction` performs case analysis and induction following the definition of a function. It makes use of a principle generated by `Function` (see Section 2.3) or `Functional Scheme` (see Section 8.15).

```coq
Coq < Functional Scheme minus_ind := Induction for minus Sort Prop.
minus_equation is defined
minus_ind is defined

Coq <
Coq < Lemma le_minus : forall n m:nat, (n - m <= n).
1 subgoal

    ============================
    forall n m : nat, n - m <= n

Coq < intros n m.
1 subgoal

    n : nat
m : nat

    ============================
    n - m <= n

Coq < functional induction (minus n m); simpl; auto.
Proof completed.

Coq < Qed.
```

**Remark:** `(qualid term_1 \ldots term_n)` must be a correct full application of `qualid`. In particular, the rules for implicit arguments are the same as usual. For example use `@qualid` if you want to write implicit arguments explicitly.

**Remark:** Parenthesis over `qualid \ldots term_n` are mandatory.

**Remark:** `functional induction (f x1 x2 x3)` is actually a wrapper for `induction x1 x2 x3 (f x1 x2 x3)` using `qualid` followed by a cleaning phase, where `qualid` is the induction principle registered for `f` (by the `Function` (see Section 2.3) or `Functional Scheme` (see Section 8.15) command) corresponding to the sort of the goal. Therefore `functional induction` may fail if the induction scheme (`qualid`) is not defined. See also Section 2.3 for the function terms accepted by `Function`.

**Remark:** There is a difference between obtaining an induction scheme for a function by using `Function` (see Section 2.3) and by using `Functional Scheme` after a normal definition using `Fixpoint` or `Definition`. See 2.3 for details.
8.8 Equality

See also: 2.3, 8.15, 10.4, 8.10.3

Error messages:

1. Cannot find induction information on qualid

2. Not the right number of induction arguments

Variants:

1. functional induction (qualid term₁ ... termₙ) using termₘ₊₁ with termₙ₊₁ ... termₘ
   Similar to Induction and elim (see Section 8.7), allows to give explicitly the induction principle and the values of dependent premises of the elimination scheme, including predicates for mutual induction when qualid is part of a mutually recursive definition.

2. functional induction (qualid term₁ ... termₙ) using termₘ₊₁ with ref₁ := termₙ₊₁ ... refₘ := termₙ
   Similar to induction and elim (see Section 8.7).

3. All previous variants can be extended by the usual as intro_pattern construction, similar for example to induction and elim (see Section 8.7).

8.8 Equality

These tactics use the equality eq:forall A:Type, A->A->Prop defined in file Logic.v (see Section 3.1.2). The notation for eq T t u is simply t=u dropping the implicit type of t and u.

8.8.1 rewrite term

This tactic applies to any goal. The type of term must have the form

forall (x₁:A₁) ... (xₙ:Aₙ) eq term₁ term₂,

where eq is the Leibniz equality or a registered setoid equality.

Then rewrite term finds the first subterm matching term₁ in the goal, resulting in instances term₁' and term₂' and then replaces every occurrence of term₁' by term₂'. Hence, some of the variables xᵢ are solved by unification, and some of the types A₁, ..., Aₙ become new subgoals.

Error messages:

1. The term provided does not end with an equation

2. Tactic generated a subgoal identical to the original goal
   This happens if term₁ does not occur in the goal.

Variants:

1. rewrite -> term
   Is equivalent to rewrite term
2. **rewrite <- term**

   Uses the equality \( \text{term}_1 = \text{term}_2 \) from right to left

3. **rewrite term in clause**

   Analogous to **rewrite term** but rewriting is done following **clause** (similarly to 8.5). For instance:
   - **rewrite H in H1** will rewrite \( H \) in the hypothesis \( H1 \) instead of the current goal.
   - **rewrite H in H1 at 1, H2 at - 2 |* means rewrite H; rewrite H in H1 at 1; rewrite H in H2 at - 2. In particular a failure will happen if any of these three simpler tactics fails.**
   - **rewrite H in * |- will do rewrite H in H_i for all hypothesis H_i <> H. A success will happen as soon as at least one of these simpler tactics succeeds.**
   - **rewrite H in * is a combination of rewrite H and rewrite H in * |- that succeeds if at least one of these two tactics succeeds.**

   **Orientation -> or <- can be inserted before the term to rewrite.**

4. **rewrite term at occurrences**

   Rewrite only the given occurrences of **term'_i.** Occurrences are specified from left to right as for **pattern** (§8.5.7). The rewrite is always performed using setoid rewriting, even for Leibniz’s equality, so one has to **Import Setoid** to use this variant.

5. **rewrite term by tactic**

   Use **tactic** to completely solve the side-conditions arising from the rewrite.

6. **rewrite term_1, ..., term_n**

   Is equivalent to the **n** successive tactics **rewrite** **term_1 up to rewrite** **term_n**, each one working on the first subgoal generated by the previous one. **Orientation -> or <- can be inserted before each term to rewrite. One unique clause can be added at the end after the keyword in; it will then affect all rewrite operations.**

7. In all forms of **rewrite** described above, a term to rewrite can be immediately prefixed by one of the following modifiers:
   - **? : the tactic rewrite ?term performs the rewrite of term as many times as possible (perhaps zero time). This form never fails.**
   - **n? : works similarly, except that it will do at most **n rewrites.**
   - **! : works as ?, except that at least one rewrite should succeed, otherwise the tactic fails.**
   - **n! (or simply **n) : precisely **n rewrites of term will be done, leading to failure if these **n rewrites are not possible.**

8. **er rewrite term**

   This tactic works as **rewrite term** but turning unresolved bindings into existential variables, if any, instead of failing. It has the same variants as **rewrite** has.

**8.8.2 cut rewrite -> term_1 = term_2**

This tactic acts like replace **term_1** with **term_2** (see below).
8.8.3 replace \( \text{term}_1 \) with \( \text{term}_2 \)

This tactic applies to any goal. It replaces all free occurrences of \( \text{term}_1 \) in the current goal with \( \text{term}_2 \) and generates the equality \( \text{term}_2 = \text{term}_1 \) as a subgoal. This equality is automatically solved if it occurs amongst the assumption, or if its symmetric form occurs. It is equivalent to cut \( \text{term}_2 = \text{term}_1 \); [intro \( \text{H} \); rewrite \( <- \text{H} \); clear \( \text{H} \); assumption || symmetry; try assumption].

Error messages:

1. terms do not have convertible types

Variants:

1. replace \( \text{term}_1 \) with \( \text{term}_2 \) by tactic

   This acts as replace \( \text{term}_1 \) with \( \text{term}_2 \) but applies tactic to solve the generated subgoal \( \text{term}_2 = \text{term}_1 \).

2. replace \( \text{term} \)

   Replace \( \text{term} \) with \( \text{term}' \) using the first assumption whose type has the form \( \text{term} = \text{term}' \) or \( \text{term}' = \text{term} \)

3. replace \( \rightarrow \) \( \text{term} \)

   Replace \( \text{term} \) with \( \text{term}' \) using the first assumption whose type has the form \( \text{term} = \text{term}' \)

4. replace \( \leftarrow \) \( \text{term} \)

   Replace \( \text{term} \) with \( \text{term}' \) using the first assumption whose type has the form \( \text{term}' = \text{term} \)

5. replace \( \text{term}_1 \) with \( \text{term}_2 \) clause

   replace \( \text{term}_1 \) with \( \text{term}_2 \) clause by tactic
   replace \( \text{term} \) clause
   replace \( \rightarrow \) \( \text{term} \) clause
   replace \( \leftarrow \) \( \text{term} \) clause

   Act as before but the replacements take place in clause (see Section 8.5) and not only in the conclusion of the goal.

   The clause argument must not contain any type of nor value of.

8.8.4 reflexivity

This tactic applies to a goal which has the form \( t = u \). It checks that \( t \) and \( u \) are convertible and then solves the goal. It is equivalent to apply refl_equal.

Error messages:

1. The conclusion is not a substitutive equation

2. Impossible to unify ... with ....

8.8.5 symmetry

This tactic applies to a goal which has the form \( t = u \) and changes it into \( u = t \).

Variant: symmetry in ident

If the statement of the hypothesis ident has the form \( t = u \), the tactic changes it to \( u = t \).
8.8.6 transitivity term

This tactic applies to a goal which has the form \( t = u \) and transforms it into the two subgoals \( t = \text{term} \) and \( \text{term} = u \).

8.8.7 subst ident

This tactic applies to a goal which has \( \text{ident} \) in its context and (at least) one hypothesis, say \( \text{H} \), of type \( \text{ident} = t \) or \( t = \text{ident} \). Then it replaces \( \text{ident} \) by \( t \) everywhere in the goal (in the hypotheses and in the conclusion) and clears \( \text{ident} \) and \( \text{H} \) from the context.

Remark: When several hypotheses have the form \( \text{ident} = t \) or \( t = \text{ident} \), the first one is used.

Variants:

1. subst \( \text{ident}_1 \ldots \text{ident}_n \)
   Is equivalent to subst \( \text{ident}_1; \ldots; \text{subst} \text{ident}_n \).

2. subst
   Applies subst repeatedly to all identifiers from the context for which an equality exists.

8.8.8 stepl term

This tactic is for chaining rewriting steps. It assumes a goal of the form \( "R \text{term}_1 \text{term}_2" \) where \( R \) is a binary relation and relies on a database of lemmas of the form \( \forall x y z, R x y \rightarrow eq x z \rightarrow R z y \) where \( eq \) is typically a setoid equality. The application of \( \text{stepl term} \) then replaces the goal by \( "R \text{term term}_2" \) and adds a new goal stating \( "eq \text{term term}_1" \).

Lemmas are added to the database using the command

\[ \text{Declare Left Step term} \. \]

The tactic is especially useful for parametric setoids which are not accepted as regular setoids for \( \text{rewrite} \) and \( \text{setoid_replace} \) (see Chapter 24).

Variants:

1. stepl term by tactic
   This applies stepl term then applies tactic to the second goal.

2. stepr term
   stepr term by tactic
   This behaves as stepl but on the right-hand-side of the binary relation. Lemmas are expected to be of the form \( \forall x y z, R x y \rightarrow eq y z \rightarrow R x z \)" and are registered using the command

\[ \text{Declare Right Step term} \. \]

8.8.9 \( \_f\_\text{equal} \)

This tactic applies to a goal of the form \( f \ a_1 \ldots a_n = f' \ a'_1 \ldots a'_n \). Using \( \_f\_\text{equal} \) on such a goal leads to subgoals \( f = f' \) and \( a_1 = a'_1 \) and so on up to \( a_n = a'_n \). Amongst these subgoals, the simple ones (e.g. provable by reflexivity or congruence) are automatically solved by \( \_f\_\text{equal} \).
8.9 Equality and inductive sets

We describe in this section some special purpose tactics dealing with equality and inductive sets or types. These tactics use the equality eq:forall (A:Type), A->A->Prop, simply written with the infix symbol =.

8.9.1 decide equality

This tactic solves a goal of the form forall x y : R, {x=y}+{~x=y}, where R is an inductive type such that its constructors do not take proofs or functions as arguments, nor objects in dependent types.

Variants:

1. decide equality term1 term2.
   Solves a goal of the form {term1=term2}+{~term1=term2}.

8.9.2 compare term1 term2

This tactic compares two given objects term1 and term2 of an inductive datatype. If G is the current goal, it leaves the sub-goals term1=term2 -> G and ~term1=term2 -> G. The type of term1 and term2 must satisfy the same restrictions as in the tactic decide equality.

8.9.3 discriminate term

This tactic proves any goal from an assumption stating that two structurally different terms of an inductive set are equal. For example, from (S (S O))=(S O) we can derive by absurdity any proposition.

   The argument term is assumed to be a proof of a statement of conclusion term1 = term2 with term1 and term2 being elements of an inductive set. To build the proof, the tactic traverses the normal forms of term1 and term2 looking for a couple of subterms u and w (u subterm of the normal form of term1 and w subterm of the normal form of term2), placed at the same positions and whose head symbols are two different constructors. If such a couple of subterms exists, then the proof of the current goal is completed, otherwise the tactic fails.

Remark: The syntax discriminate ident can be used to refer to a hypothesis quantified in the goal. In this case, the quantified hypothesis whose name is ident is first introduced in the local context using intros until ident.

Error messages:

1. No primitive equality found
2. Not a discriminable equality

Variants:

1. discriminate num
   This does the same thing as intros until num followed by discriminate ident where ident is the identifier for the last introduced hypothesis.

Reminder: opaque constants will not be expanded by δ reductions
2. **discriminate** term with bindings_list

This does the same thing as **discriminate** term but using the given bindings to instantiate parameters or hypotheses of term.

3. **ediscriminate** num
   ediscriminate term [with bindings_list]

This works the same as **discriminate** but if the type of term, or the type of the hypothesis referred to by num, has uninstatiable parameters, these parameters are left as existential variables.

4. **discriminate**

This behaves like **discriminate** ident if ident is the name of an hypothesis to which **discriminate** is applicable; if the current goal is of the form term₁ <> term₂, this behaves as intro ident; injection ident.

**Error messages:**

(a) No discriminable equalities occurs when the goal does not verify the expected preconditions.

### 8.9.4 **injection** term

The **injection** tactic is based on the fact that constructors of inductive sets are injections. That means that if c is a constructor of an inductive set, and if (c t₁₁) and (c t₂₂) are two terms that are equal then t₁ and t₂ are equal too.

If term is a proof of a statement of conclusion term₁ = term₂, then **injection** applies injectivity as deep as possible to derive the equality of all the subterms of term₁ and term₂ placed in the same positions. For example, from (S (S n))=(S (S (S m)) we may derive n=(S m). To use this tactic term₁ and term₂ should be elements of an inductive set and they should be neither explicitly equal, nor structurally different. We mean by this that, if n₁ and n₂ are their respective normal forms, then:

- n₁ and n₂ should not be syntactically equal,
- there must not exist any pair of subterms u and w, u subterm of n₁ and w subterm of n₂, placed in the same positions and having different constructors as head symbols.

If these conditions are satisfied, then, the tactic derives the equality of all the subterms of term₁ and term₂ placed in the same positions and puts them as antecedents of the current goal.

**Example:** Consider the following goal:

Coq < Inductive list : Set :=
Coq < | nil : list
Coq < | cons : nat -> list -> list.
Coq < Variable P : list -> Prop.
Coq < Show.
1 subgoal
l : list
n : nat
\[ H : P \text{ nil} \]
\[ H_0 : \text{cons } n \text{ l = cons } 0 \text{ nil} \]
\[ \text{=============} \]
\[ P \text{ l} \]

Coq < injection H0.
1 subgoal

\[ l : \text{list} \]
\[ n : \text{nat} \]
\[ H : P \text{ nil} \]
\[ H_0 : \text{cons } n \text{ l = cons } 0 \text{ nil} \]
\[ \text{=============} \]
\[ l = \text{nil } \rightarrow n = 0 \rightarrow P \text{ l} \]

Beware that injection yields always an equality in a sigma type whenever the injected object has a dependent type.

**Remark:** There is a special case for dependent pairs. If we have a decidable equality over the type of the first argument, then it is safe to do the projection on the second one, and so injection will work fine. To define such an equality, you have to use the Scheme command (see 8.14).

**Remark:** If some quantified hypothesis of the goal is named \textit{ident}, then injection \textit{ident} first introduces the hypothesis in the local context using intros until \textit{ident}.

**Error messages:**

1. Not a projectable equality but a discriminable one
2. Nothing to do, it is an equality between convertible terms
3. Not a primitive equality

**Variants:**

1. injection \textit{num}
   
   This does the same thing as intros until \textit{num} followed by injection \textit{ident} where \textit{ident} is the identifier for the last introduced hypothesis.

2. injection \textit{term} with bindings_list
   
   This does the same as injection \textit{term} but using the given bindings to instantiate parameters or hypotheses of \textit{term}.

3. einjection \textit{num}
   einjection \textit{term [with bindings_list]}
   
   This works the same as injection but if the type of \textit{term}, or the type of the hypothesis referred to by \textit{num}, has uninstatiated parameters, these parameters are left as existential variables.

4. injection
   
   If the current goal is of the form \textit{term}_1 \text{<>} \textit{term}_2, this behaves as intro \textit{ident}; injection \textit{ident}.

**Error message:** goal does not satisfy the expected preconditions
5. injection term \([\text{with } \text{bindings}_\text{list}]\) as intro_pattern \(\ldots\) intro_pattern
   injection num as intro_pattern \(\ldots\) intro_pattern
   einjection term \([\text{with } \text{bindings}_\text{list}]\) as intro_pattern \(\ldots\) intro_pattern
   einjection num as intro_pattern \(\ldots\) intro_pattern
   einjection as intro_pattern \(\ldots\) intro_pattern

These variants apply intros intro_pattern \(\ldots\) intro_pattern after the call to injection or einjection.

8.9.5 simplify_eq term

Let term be the proof of a statement of conclusion \(\text{term}_1 = \text{term}_2\). If \(\text{term}_1\) and \(\text{term}_2\) are structurally different (in the sense described for the tactic discriminate), then the tactic simplify_eq behaves as discriminate term, otherwise it behaves as injection term.

Remark: If some quantified hypothesis of the goal is named ident, then simplify_eq ident first introduces the hypothesis in the local context using intros until ident.

Variants:

1. simplify_eq num
   This does the same thing as intros until num then simplify_eq ident where ident is the identifier for the last introduced hypothesis.

2. simplify_eq term with bindings_list
   This does the same as simplify_eq term but using the given bindings to instantiate parameters or hypotheses of term.

3. esimplify_eq num
   esimplify_eq term \([\text{with } \text{bindings}_\text{list}]\)
   This works the same as simplify_eq but if the type of term, or the type of the hypothesis referred to by num, has uninstantiated parameters, these parameters are left as existential variables.

4. simplify_eq
   If the current goal has form \(t_1 \leftrightarrow t_2\), it behaves as intro ident; simplify_eq ident.

8.9.6 dependent rewrite \(\rightarrow\) ident

This tactic applies to any goal. If ident has type \((\text{existT } B \ a \ b) = (\text{existT } B \ a' \ b')\) in the local context (i.e. each term of the equality has a sigma type \(\{a : A \& (B \ a)\}\)) this tactic rewrites \(a\) into \(a'\) and \(b\) into \(b'\) in the current goal. This tactic works even if \(B\) is also a sigma type. This kind of equalities between dependent pairs may be derived by the injection and inversion tactics.

Variants:

1. dependent rewrite \(\leftarrow\) ident
   Analogous to dependent rewrite \(\rightarrow\) but uses the equality from right to left.
8.10 Inversion

8.10.1 inversion ident

Let the type of ident in the local context be \((I \vec{t})\), where \(I\) is a (co)inductive predicate. Then, inversion applied to ident derives for each possible constructor \(c_i\) of \((I \vec{t})\), all the necessary conditions that should hold for the instance \((I \vec{t})\) to be proved by \(c_i\).

Remark: If ident does not denote a hypothesis in the local context but refers to a hypothesis quantified in the goal, then the latter is first introduced in the local context using intros until ident.

Variants:

1. inversion num
   
   This does the same thing as intros until num then inversion ident where ident is the identifier for the last introduced hypothesis.

2. inversion_clear ident
   
   This behaves as inversion and then erases ident from the context.

3. inversion ident as intro_pattern
   
   This behaves as inversion but using names in intro_pattern for naming hypotheses. The intro_pattern must have the form \([p_1 \ldots p_{n_1} \mid \ldots \mid p_m \ldots p_{n_m}]\) with \(m\) being the number of constructors of the type of ident. Be careful that the list must be of length \(m\) even if inversion discards some cases (which is precisely one of its roles): for the discarded cases, just use an empty list (i.e. \(n_i = 0\)).

   The arguments of the \(i^{th}\) constructor and the equalities that inversion introduces in the context of the goal corresponding to the \(i^{th}\) constructor, if it exists, get their names from the list \(p_1 \ldots p_{n_i}\) in order. If there are not enough names, induction invents names for the remaining variables to introduce. In case an equation splits into several equations (because inversion applies injection on the equalities it generates), the corresponding name \(p_{ij}\) in the list must be replaced by a sublist of the form \([p_{ij1} \ldots p_{ijq}]\) (or, equivalently, \((p_{ij1}, \ldots, p_{ijq})\)) where \(q\) is the number of subequalities obtained from splitting the original equation. Here is an example.

```coq
Coq < Inductive contains0 : list nat -> Prop :=
Coq <   | in_hd : forall l, contains0 (0 :: l)
Coq <   | in_tl : forall l b, contains0 l -> contains0 (b :: l).
contains0 is defined
contains0_ind is defined

Coq < Goal forall l:list nat, contains0 (1 :: l) -> contains0 l.
1 subgoal

=============
forall l : list nat, contains0 (1 :: l) -> contains0 l

Coq < intros l H; inversion H as [ | l' p Hl' [Heqp Heql'] ].
1 subgoal

    l : list nat
    H : contains0 (1 :: l)
```
l' : list nat
p : nat
Hl' : contains0 l
Heqp : p = 1
Heql' : l' = l

\begin{verbatim}
\textbf{4. inversion num as intro_pattern}
\end{verbatim}

This allows to name the hypotheses introduced by \texttt{inversion num} in the context.

\begin{verbatim}
\textbf{5. inversion\_clear ident as intro_pattern}
\end{verbatim}

This allows to name the hypotheses introduced by \texttt{inversion\_clear} in the context.

\begin{verbatim}
\textbf{6. inversion ident in ident_1 \ldots \ ident_n}
\end{verbatim}

Let \texttt{ident_1 \ldots ident_n}, be identifiers in the local context. This tactic behaves as generalizing \texttt{ident_1 \ldots ident_n}, and then performing \texttt{inversion}.

\begin{verbatim}
\textbf{7. inversion\_clear ident as intro_pattern in ident_1 \ldots \ ident_n}
\end{verbatim}

This allows to name the hypotheses introduced in the context by \texttt{inversion\_clear ident in ident_1 \ldots ident_n}.

\begin{verbatim}
\textbf{8. inversion\_clear\_as ident in ident_1 \ldots \ ident_n}
\end{verbatim}

Let \texttt{ident_1 \ldots ident_n}, be identifiers in the local context. This tactic behaves as generalizing \texttt{ident_1 \ldots ident_n}, and then performing \texttt{inversion\_clear}.

\begin{verbatim}
\textbf{9. inversion\_clear\_as intro_pattern in ident_1 \ldots \ ident_n}
\end{verbatim}

This allows to name the hypotheses introduced in the context by \texttt{inversion\_clear\_as intro_pattern in ident_1 \ldots ident_n}.

\begin{verbatim}
\textbf{10. dependent inversion ident}
\end{verbatim}

That must be used when \texttt{ident} appears in the current goal. It acts like \texttt{inversion} and then substitutes \texttt{ident} for the corresponding term in the goal.

\begin{verbatim}
\textbf{11. dependent inversion\_as intro_pattern}
\end{verbatim}

This allows to name the hypotheses introduced in the context by \texttt{dependent inversion\_as intro_pattern}.

\begin{verbatim}
\textbf{12. dependent inversion\_clear ident}
\end{verbatim}

Like \texttt{dependent inversion}, except that \texttt{ident} is cleared from the local context.

\begin{verbatim}
\textbf{13. dependent inversion\_clear\_as intro_pattern}
\end{verbatim}

This allows to name the hypotheses introduced in the context by \texttt{dependent inversion\_clear\_as intro_pattern}.

\begin{verbatim}
\textbf{14. dependent inversion ident with term}
\end{verbatim}

This variant allows you to specify the generalization of the goal. It is useful when the system fails to generalize the goal automatically. If \texttt{ident} has type \((I \overrightarrow{t})\) and \(I\) has type \texttt{forall(\overrightarrow{x} : \overrightarrow{T})}, \(s\), then \texttt{term} must be of type \(I : \texttt{forall(\overrightarrow{x} : \overrightarrow{T})}, I \overrightarrow{x} \rightarrow s'\) where \(s'\) is the type of the goal.
15. dependent inversion ident as intro_pattern with term

This allows to name the hypotheses introduced in the context by dependent inversion ident with term.

16. dependent inversion_clear ident with term

Like dependent inversion ... with but clears ident from the local context.

17. dependent inversion_clear ident as intro_pattern with term

This allows to name the hypotheses introduced in the context by dependent inversion_clear ident with term.

18. simple inversion ident

It is a very primitive inversion tactic that derives all the necessary equalities but it does not simplify the constraints as inversion does.

19. simple inversion ident as intro_pattern

This allows to name the hypotheses introduced in the context by simple inversion.

20. inversion ident using ident'

Let ident have type \((I \overline{t})\) (I an inductive predicate) in the local context, and ident' be a (dependent) inversion lemma. Then, this tactic refines the current goal with the specified lemma.

21. inversion ident using ident' in ident_1... ident_n

This tactic behaves as generalizing ident_1... ident_n, then doing inversion ident using ident'.

See also: 10.5 for detailed examples

8.10.2 Derive Inversion ident with forall(\overline{x}:T), I \overline{t} Sort sort

This command generates an inversion principle for the inversion ... using tactic. Let I be an inductive predicate and \(\overline{x}\) the variables occurring in \(\overline{t}\). This command generates and stocks the inversion lemma for the sort sort corresponding to the instance forall(\overline{x}:T), I \overline{t} with the name ident in the global environment. When applied it is equivalent to have inverted the instance with the tactic inversion.

Variants:

1. Derive Inversion_clear ident with forall(\overline{x}:T), I \overline{t} Sort sort

When applied it is equivalent to having inverted the instance with the tactic inversion replaced by the tactic inversion_clear.

2. Derive Dependent Inversion ident with forall(\overline{x}:T), I \overline{t} Sort sort

When applied it is equivalent to having inverted the instance with the tactic dependent inversion.

3. Derive Dependent Inversion_clear ident with forall(\overline{x}:T), I \overline{t} Sort sort

When applied it is equivalent to having inverted the instance with the tactic dependent inversion_clear.

See also: 10.5 for examples
8.10.3 functional inversion *ident*

functional inversion is a *highly* experimental tactic which performs inversion on hypothesis *ident* of the form \textit{qualid \ term}_1 \ldots \textit{term}_n \ = \ \textit{term} or \textit{term} = \textit{qualid \ term}_1 \ldots \textit{term}_n where \textit{qualid} must have been defined using \textit{Function} (see Section 2.3).

**Error messages:**

1. Hypothesis *ident* must contain at least one Function

2. Cannot find inversion information for hypothesis *ident* This error may be raised when some inversion lemma failed to be generated by Function.

**Variants:**

1. functional inversion *num*
   
   This does the same thing as intros until *num* then functional inversion *ident* where *ident* is the identifier for the last introduced hypothesis.

2. functional inversion *ident qualid*

   functional inversion *num qualid*

   In case the hypothesis *ident* (or *num*) has a type of the form \textit{qualid}_1 \ \textit{term}_1 \ldots \textit{term}_n \ = \ \textit{qualid}_2 \ \textit{term}_{n+1} \ldots \textit{term}_{n+m} where \textit{qualid}_1 and \textit{qualid}_2 are valid candidates to functional inversion, this variant allows to choose which must be inverted.

8.10.4 *quote* *ident*

This kind of inversion has nothing to do with the tactic inversion above. This tactic does change \textit{(ident t)}, where \textit{t} is a term built in order to ensure the convertibility. In other words, it does inversion of the function *ident*. This function must be a fixpoint on a simple recursive datatype: see 10.8 for the full details.

**Error messages:**

1. *quote*: not a simple fixpoint

   Happens when *quote* is not able to perform inversion properly.

**Variants:**

1. *quote* *ident* [ *ident_1 \ldots ident_n* ]

   All terms that are built only with \textit{ident}_1 \ldots \textit{ident}_n will be considered by *quote* as constants rather than variables.

8.11 Classical tactics

In order to ease the proving process, when the Classical module is loaded. A few more tactics are available. Make sure to load the module using the Require Import command.
8.11.1 classical_left, classical_right

The tactics classical_left and classical_right are the analog of the left and right but using classical logic. They can only be used for disjunctions. Use classical_left to prove the left part of the disjunction with the assumption that the negation of right part holds. Use classical_right to prove the right part of the disjunction with the assumption that the negation of left part holds.

8.12 Automatizing

8.12.1 auto

This tactic implements a Prolog-like resolution procedure to solve the current goal. It first tries to solve the goal using the assumption tactic, then it reduces the goal to an atomic one using intros and introducing the newly generated hypotheses as hints. Then it looks at the list of tactics associated to the head symbol of the goal and tries to apply one of them (starting from the tactics with lower cost). This process is recursively applied to the generated subgoals.

By default, auto only uses the hypotheses of the current goal and the hints of the database named core.

Variants:

1. auto num
   Forces the search depth to be num. The maximal search depth is 5 by default.

2. auto with ident_1 ... ident_n
   Uses the hint databases ident_1 ... ident_n in addition to the database core. See Section 8.13.1 for the list of pre-defined databases and the way to create or extend a database. This option can be combined with the previous one.

3. auto with *
   Uses all existing hint databases, minus the special database v62. See Section 8.13.1

4. auto using lemma_1, ..., lemma_n
   Uses lemma_1, ..., lemma_n in addition to hints (can be combined with the with ident option).

5. trivial
   This tactic is a restriction of auto that is not recursive and tries only hints which cost 0. Typically it solves trivial equalities like \(X = X\).

6. trivial with ident_1 ... ident_n

7. trivial with *

Remark: auto either solves completely the goal or else leaves it intact. auto and trivial never fail.

See also: Section 8.13.1
8.12.2  eauto

This tactic generalizes auto. In contrast with the latter, eauto uses unification of the goal against the hints rather than pattern-matching (in other words, it uses eapply instead of apply). As a consequence, eauto can solve such a goal:

Coq < Hint Resolve ex_intro.
Warning: the hint: eapply ex_intro will only be used by eauto
Coq < Goal forall P:nat -> Prop, P 0 -> exists n, P n.
1 subgoal

============================
 forall P0 : nat -> Prop, P0 0 -> exists n : nat, P0 n

Coq < eauto.
Proof completed.

Note that ex_intro should be declared as an hint.

See also: Section 8.13.1

8.12.3  tauto

This tactic implements a decision procedure for intuitionistic propositional calculus based on the contraction-free sequent calculi LJT* of Roy Dyckhoff [54]. Note that tauto succeeds on any instance of an intuitionistic tautological proposition. tauto unfolds negations and logical equivalence but does not unfold any other definition.

The following goal can be proved by tauto whereas auto would fail:

Coq < Goal forall (x:nat) (P:nat -> Prop), x = 0 / P x -> x <> 0 -> P x.
1 subgoal

============================
 forall (x : nat) (P0 : nat -> Prop), x = 0 / P0 x -> x <> 0 -> P0 x

Coq < intros.
1 subgoal

x : nat
P0 : nat -> Prop
H : x = 0 / P0 x
H0 : x <> 0

============================
 P0 x

Coq <  tauto.
Proof completed.

Moreover, if it has nothing else to do, tauto performs introductions. Therefore, the use of intros in the previous proof is unnecessary. tauto can for instance prove the following:

Coq < (* auto would fail *)
Coq < Goal forall (A:Prop) (P:nat -> Prop),
Coq <     A \ (forall x:nat, ~ A -> P x) -> forall x:nat, ~ A -> P x.
8.12 Automatizing

1 subgoal

\[
\forall (A : \text{Prop}) (P0 : \text{nat} \rightarrow \text{Prop}),
A \lor (\forall x : \text{nat}, \neg A \rightarrow P0 x) \rightarrow \forall x : \text{nat}, \neg A \rightarrow P0 x
\]

Coq <
Coq < tauto.
Proof completed.

**Remark:** In contrast, tauto cannot solve the following goal

Coq < Goal \forall (A:Prop) (P:nat -> Prop),
Coq < A \lor (forall x:nat, ~ A -> P x) -> forall x:nat, ~ ~ (A \lor P x).

because (forall x:nat, ~ A -> P x) cannot be treated as atomic and an instantiation of x is necessary.

8.12.4 intuition tactic

The tactic intuition takes advantage of the search-tree built by the decision procedure involved in the tactic tauto. It uses this information to generate a set of subgoals equivalent to the original one (but simpler than it) and applies the tactic tactic to them [103]. If this tactic fails on some goals then intuition fails. In fact, tauto is simply intuition fail.

For instance, the tactic intuition auto applied to the goal

\[(\forall (x:nat), P x)/\lor B \rightarrow (\forall (y:nat), P y)/\lor P O \lor/\lor P O\]

internally replaces it by the equivalent one:

\[(\forall (x:nat), P x), B |- P O\]

and then uses auto which completes the proof.

Originally due to César Muñoz, these tactics (tauto and intuition) have been completely re-engineered by David Delahaye using mainly the tactic language (see Chapter 9). The code is now much shorter and a significant increase in performance has been noticed. The general behavior with respect to dependent types, unfolding and introductions has slightly changed to get clearer semantics. This may lead to some incompatibilities.

**Variants:**

1. intuition
   Is equivalent to intuition auto with *.

8.12.5 rtauto

The rtauto tactic solves propositional tautologies similarly to what tauto does. The main difference is that the proof term is built using a reflection scheme applied to a sequent calculus proof of the goal. The search procedure is also implemented using a different technique.

Users should be aware that this difference may result in faster proof-search but slower proof-checking, and rtauto might not solve goals that tauto would be able to solve (e.g. goals involving universal quantifiers).
8.12.6 firstorder

The tactic firstorder is an experimental extension of tauto to first-order reasoning, written by Pierre Corbineau. It is not restricted to usual logical connectives but instead may reason about any first-order class inductive definition.

Variants:

1. firstorder tactic
   
   Tries to solve the goal with tactic when no logical rule may apply.

2. firstorder with ident₁ … identₙ
   
   Adds lemmas ident₁ … identₙ to the proof-search environment.

3. firstorder using ident₁ … identₙ
   
   Adds lemmas in auto hints bases ident₁ … identₙ to the proof-search environment.

Proof-search is bounded by a depth parameter which can be set by typing the Set Firstorder Depth n vernacular command.

8.12.7 congruence

The tactic congruence, by Pierre Corbineau, implements the standard Nelson and Oppen congruence closure algorithm, which is a decision procedure for ground equalities with uninterpreted symbols. It also includes the constructor theory (see 8.9.4 and 8.9.3). If the goal is a non-quantified equality, congruence tries to prove it with non-quantified equalities in the context. Otherwise it tries to infer a discriminable equality from those in the context. Alternatively, congruence tries to prove that a hypothesis is equal to the goal or to the negation of another hypothesis.

congruence is also able to take advantage of hypotheses stating quantified equalities, you have to provide a bound for the number of extra equalities generated that way. Please note that one of the members of the equality must contain all the quantified variables in order for congruence to match against it.

Coq < Theorem T:
Coq < a=(f a) -> (g b (f a))=(f (f a)) -> (g a b)=(f (g b a)) -> (g a b)=a.
1 subgoal

============================
 a = f a -> g b (f a) = f (f a) -> g a b = f (g b a) -> g a b = a
Coq < intros.
1 subgoal

H : a = f a
H0 : g b (f a) = f (f a)
H1 : g a b = f (g b a)

============================
 g a b = a
Coq < congruence.
Proof completed.
8.12 Automatizing

Coq < Theorem inj : f = pair a -> Some (f c) = Some (f d) -> c=d.
1 subgoal

============================
 f = pair a -> Some (f c) = Some (f d) -> c = d

Coq < intros.
1 subgoal

H : f = pair a
H0 : Some (f c) = Some (f d)

============================
 c = d

Coq < congruence.
Proof completed.

Variants:

1. congruence \( n \)
   
   Tries to add at most \( n \) instances of hypotheses stating quantified equalities to the problem in order to solve it. A bigger value of \( n \) does not make success slower, only failure. You might consider adding some lemmas as hypotheses using assert in order for congruence to use them.

Variants:

1. congruence with \( \text{term}_1 \ldots \text{term}_n \)

   Adds \( \text{term}_1 \ldots \text{term}_n \) to the pool of terms used by congruence. This helps in case you have partially applied constructors in your goal.

Error messages:

1. I don’t know how to handle dependent equality
   
   The decision procedure managed to find a proof of the goal or of a discriminable equality but this proof couldn’t be built in Coq because of dependently-typed functions.

2. I couldn’t solve goal
   
   The decision procedure didn’t find any way to solve the goal.

3. Goal is solvable by congruence but some arguments are missing.
   
   Try "congruence with ...", replacing metavariables by arbitrary terms.
   
   The decision procedure could solve the goal with the provision that additional arguments are supplied for some partially applied constructors. Any term of an appropriate type will allow the tactic to successfully solve the goal. Those additional arguments can be given to congruence by filling in the holes in the terms given in the error message, using the with variant described above.
8.12.8 omega

The tactic omega, due to Pierre Crégut, is an automatic decision procedure for Presburger arithmetic. It solves quantifier-free formulas built with $\sim$, $\land$, $\lor$, $\rightarrow$ on top of equalities, inequalities and disequalities on both the type $\text{nat}$ of natural numbers and $\mathbb{Z}$ of binary integers. This tactic must be loaded by the command Require Import Omega. See the additional documentation about omega (see Chapter 19).

8.12.9 ring and ring_simplify term$_1$ ... term$_n$

The ring tactic solves equations upon polynomial expressions of a ring (or semi-ring) structure. It proceeds by normalizing both hand sides of the equation (w.r.t. associativity, commutativity and distributivity, constant propagation) and comparing syntactically the results.

ring_simplify applies the normalization procedure described above to the terms given. The tactic then replaces all occurrences of the terms given in the conclusion of the goal by their normal forms. If no term is given, then the conclusion should be an equation and both hand sides are normalized.

See Chapter 23 for more information on the tactic and how to declare new ring structures.

8.12.10 field, field_simplify term$_1$... term$_n$ and field_simplify_eq

The field tactic is built on the same ideas as ring: this is a reflexive tactic that solves or simplifies equations in a field structure. The main idea is to reduce a field expression (which is an extension of ring expressions with the inverse and division operations) to a fraction made of two polynomial expressions.

Tactic field is used to solve subgoals, whereas field_simplify term$_1$...term$_n$ replaces the provided terms by their reduced fraction. field_simplify_eq applies when the conclusion is an equation: it simplifies both hand sides and multiplies so as to cancel denominators. So it produces an equation without division nor inverse.

All of these 3 tactics may generate a subgoal in order to prove that denominators are different from zero.

See Chapter 23 for more information on the tactic and how to declare new field structures.

Example:

Coq < Require Import Reals.
Coq < Goal forall x y:R,
Coq < (x * y > 0)%R ->
Coq < (x * (1 / x + x / (x + y)))%R =
Coq < ((- 1 / y) * y * (- x * (x / (x + y)) - 1))%R.

Coq < intros; field.
1 subgoal

$\text{x : R}$
$\text{y : R}$
$\text{H : (x * y > 0)%R}$

$\text{================================}$
$\text{(x + y)%R <> 0%R /\ y <> 0%R /\ x <> 0%R}$

See also: file contrib/setoid_ring/RealField.v for an example of instantiation, theory theories/Reals for many examples of use of field.
8.12 Automatizing

8.12.11 fourier

This tactic written by Loïc Pottier solves linear inequalities on real numbers using Fourier’s method [63]. This tactic must be loaded by Require Import Fourier.

Example:

Coq < Require Import Reals.
Coq < Require Import Fourier.
Coq < Goal forall x y:R, (x < y)%R -> (y + 1 >= x - 1)%R.
Coq < intros; fourier.
Proof completed.

8.12.12 autorewrite with ident_1 ... ident_n.

This tactic^5^ carries out rewritings according the rewriting rule bases ident_1 ... ident_n.

Each rewriting rule of a base ident_i is applied to the main subgoal until it fails. Once all the rules have been processed, if the main subgoal has progressed (e.g., if it is distinct from the initial main goal) then the rules of this base are processed again. If the main subgoal has not progressed then the next base is processed. For the bases, the behavior is exactly similar to the processing of the rewriting rules.

The rewriting rule bases are built with the Hint Rewrite vernacular command.

Warning: This tactic may loop if you build non terminating rewriting systems.

Variant:

1. autorewrite with ident_1 ... ident_n using tactic
   Performs, in the same way, all the rewritings of the bases ident_1 ... ident_n applying tactic to the main subgoal after each rewriting step.

2. autorewrite with ident_1 ... ident_n in qualid
   Performs all the rewritings in hypothesis qualid.

3. autorewrite with ident_1 ... ident_n in qualid using tactic
   Performs all the rewritings in hypothesis qualid applying tactic to the main subgoal after each rewriting step.

4. autorewrite with ident_1 ... ident_n in clause Performs all the rewritings in the clause clause.
   The clause argument must not contain any type of nor value of.

See also: Section 8.13.4 for feeding the database of lemmas used by autorewrite.

See also: Section 10.7 for examples showing the use of this tactic.

---

^5^The behavior of this tactic has much changed compared to the versions available in the previous distributions (V6). This may cause significant changes in your theories to obtain the same result. As a drawback of the re-engineering of the code, this tactic has also been completely revised to get a very compact and readable version.
8.13 Controlling automation

8.13.1 The hints databases for auto and eauto

The hints for auto and eauto are stored in databases. Each database maps head symbols to a list of hints. One can use the command Print Hint ident to display the hints associated to the head symbol ident (see 8.13.3). Each hint has a cost that is an nonnegative integer, and an optional pattern. The hints with lower cost are tried first. A hint is tried by auto when the conclusion of the current goal matches its pattern or when it has no pattern.

Creating Hint databases

One can optionally declare a hint database using the command Create HintDb. If a hint is added to an unknown database, it will be automatically created.

```
Create HintDb ident [discriminated]
```

This command creates a new database named ident. The database is implemented by a Discrimination Tree (DT) that serves as an index of all the lemmas. The DT can use transparency information to decide if a constant should be indexed or not (c.f. 8.13.1), making the retrieval more efficient. The legacy implementation (the default one for new databases) uses the DT only on goals without existentials (i.e., auto goals), for non-Immediate hints and do not make use of transparency hints, putting more work on the unification that is run after retrieval (it keeps a list of the lemmas in case the DT is not used). The new implementation enabled by the discriminated option makes use of DTs in all cases and takes transparency information into account. However, the order in which hints are retrieved from the DT may differ from the order in which they were inserted, making this implementation observationaly different from the legacy one.

Variants:

1. Local Hint hint_definition : ident₁ ... identₙ

This is used to declare a hint database that must not be exported to the other modules that require and import the current module. Inside a section, the option Local is useless since hints do not survive anyway to the closure of sections.

The general command to add a hint to some database ident₁, . . . , identₙ is:

```
Hint hint_definition : ident₁ ... identₙ
```

where hint_definition is one of the following expressions:

- Resolve term

  This command adds apply term to the hint list with the head symbol of the type of term. The cost of that hint is the number of subgoals generated by apply term.

  In case the inferred type of term does not start with a product the tactic added in the hint list is exact term. In case this type can be reduced to a type starting with a product, the tactic apply term is also stored in the hints list.

  If the inferred type of term contains a dependent quantification on a predicate, it is added to the hint list of eapply instead of the hint list of apply. In this case, a warning is printed since the
hint is only used by the tactic eauto (see 8.12.2). A typical example of a hint that is used only by eauto is a transitivity lemma.

**Error messages:**

1. **Bound head variable**
   
   The head symbol of the type of term is a bound variable such that this tactic cannot be associated to a constant.

2. **term cannot be used as a hint**
   
   The type of term contains products over variables which do not appear in the conclusion. A typical example is a transitivity axiom. In that case the apply tactic fails, and thus is useless.

**Variants:**

1. **Resolve term₁...termₘ**
   
   Adds each Resolve termᵢ.

- **Immediate term**

   This command adds apply term; trivial to the hint list associated with the head symbol of the type of ident in the given database. This tactic will fail if all the subgoals generated by apply term are not solved immediately by the trivial tactic (which only tries tactics with cost 0).

   This command is useful for theorems such as the symmetry of equality or \( n+1 = m+1 \rightarrow n = m \) that we may like to introduce with a limited use in order to avoid useless proof-search.

   The cost of this tactic (which never generates subgoals) is always 1, so that it is not used by trivial itself.

**Error messages:**

1. **Bound head variable**

2. **term cannot be used as a hint**

**Variants:**

1. **Immediate term₁...termₘ**

   Adds each Immediate termᵢ.

- **Constructors ident**

   If ident is an inductive type, this command adds all its constructors as hints of type Resolve. Then, when the conclusion of current goal has the form \((ident ...)\), auto will try to apply each constructor.

**Error messages:**

1. **ident is not an inductive type**

2. **ident not declared**

**Variants:**
1. Constructors \textit{ident}_1 \ldots \textit{ident}_m

   Adds each Constructors \textit{ident}_i.

- **Unfold \textit{qualid}**

  This adds the tactic \texttt{unfold qualid} to the hint list that will only be used when the head constant of the goal is \textit{ident}. Its cost is 4.

  
  **Variants:**

  1. Unfold \textit{ident}_1 \ldots \textit{ident}_m

     Adds each Unfold \textit{ident}_i.

- **Transparent,Opaque \textit{qualid}**

  This adds a transparency hint to the database, making \textit{qualid} a transparent or opaque constant during resolution. This information is used during unification of the goal with any lemma in the database and inside the discrimination network to relax or constrain it in the case of discriminated databases.

  **Variants:**

  1. Transparent,Opaque \textit{ident}_1 \ldots \textit{ident}_m

     Declares each \textit{ident}_i as a transparent or opaque constant.

- **Extern \textit{num} [\textit{pattern}] => \textit{tactic}**

  This hint type is to extend \texttt{auto} with tactics other than \texttt{apply} and \texttt{unfold}. For that, we must specify a cost, an optional pattern and a tactic to execute. Here is an example:

  \begin{verbatim}
  Hint Extern 4 ~(=?=?) => discriminate.
  \end{verbatim}

  Now, when the head of the goal is a disequality, \texttt{auto} will try \texttt{discriminate} if it does not manage to solve the goal with hints with a cost less than 4.

  One can even use some sub-patterns of the pattern in the tactic script. A sub-pattern is a question mark followed by an ident, like \texttt{?X1} or \texttt{?X2}. Here is an example:

  Coq < Require Import List.
  Coq < Hint Extern 5  ({?X1 = ?X2} + {?X1 <> ?X2}) =>
  Coq < generalize X1, X2; decide equality : eqdec.
  Coq < Goal
  Coq < forall a b:list (nat * nat), \{a = b\} + \{a <> b\}.
  1 subgoal

  ===========
  forall a b : list (nat * nat), \{a = b\} + \{a <> b\}

  Coq < info auto with eqdec.
  == intro a; intro b; generalize a, b; decide equality;
     generalize ai, p; decide equality.
     generalize bi, n0; decide equality.
  \end{verbatim}
generalize a3, n; decide equality.

Proof completed.

Remark: One can use an `Extern` hint with no pattern to do pattern-matching on hypotheses using `match goal with` inside the tactic.

Variants:

1. Hint `hint_definition`
   No database name is given: the hint is registered in the `core` database.

2. Hint Local `hint_definition : ident1 ... identn`
   This is used to declare hints that must not be exported to the other modules that require and import the current module. Inside a section, the option `Local` is useless since hints do not survive anyway to the closure of sections.

3. Hint Local `hint_definition`
   Idem for the `core` database.

### 8.13.2 Hint databases defined in the COQ standard library

Several hint databases are defined in the COQ standard library. The actual content of a database is the collection of the hints declared to belong to this database in each of the various modules currently loaded. Especially, requiring new modules potentially extend a database. At COQ startup, only the `core` and `v62` databases are non empty and can be used.

- **core** This special database is automatically used by `auto`. It contains only basic lemmas about negation, conjunction, and so on from. Most of the hints in this database come from the `Init` and `Logic` directories.

- **arith** This database contains all lemmas about Peano’s arithmetic proved in the directories `Init` and `Arith`.

- **zarith** contains lemmas about binary signed integers from the directories `theories/ZArith`. When required, the module `Omega` also extends the database `zarith` with a high-cost hint that calls `omega` on equations and inequalities in `nat` or `Z`.

- **bool** contains lemmas about booleans, mostly from directory `theories/Bool`.

- **datatypes** is for lemmas about lists, streams and so on that are mainly proved in the `Lists` subdirectory.

- **sets** contains lemmas about sets and relations from the directories `Sets` and `Relations`.

- **typeclass_instances** contains all the type class instances declared in the environment, including those used for `setoid_rewrite`, from the `Classes` directory.
There is also a special database called v62. It collects all hints that were declared in the versions of COQ prior to version 6.2.4 when the databases core, arith, and so on were introduced. The purpose of the database v62 is to ensure compatibility with further versions of COQ for developments done in versions prior to 6.2.4 (auto being replaced by auto with v62). The database v62 is intended not to be extended (!). It is not included in the hint databases list used in the auto with * tactic.

Furthermore, you are advised not to put your own hints in the core database, but use one or several databases specific to your development.

8.13.3 Print Hint

This command displays all hints that apply to the current goal. It fails if no proof is being edited, while the two variants can be used at every moment.

Variants:

1. Print Hint ident
   This command displays only tactics associated with ident in the hints list. This is independent of the goal being edited, so this command will not fail if no goal is being edited.

2. Print Hint *
   This command displays all declared hints.

3. Print HintDb ident
   This command displays all hints from database ident.

8.13.4 Hint Rewrite term_1 ... term_n : ident

This vernacular command adds the terms term_1 ... term_n (their types must be equalities) in the rewriting base ident with the default orientation (left to right). Notice that the rewriting bases are distinct from the auto hint bases and that auto does not take them into account.

This command is synchronous with the section mechanism (see 2.4): when closing a section, all aliases created by Hint Rewrite in that section are lost. Conversely, when loading a module, all Hint Rewrite declarations at the global level of that module are loaded.

Variants:

1. Hint Rewrite -> term_1 ... term_n : ident
   This is strictly equivalent to the command above (we only make explicit the orientation which otherwise defaults to -

2. Hint Rewrite <- term_1 ... term_n : ident
   Adds the rewriting rules term_1 ... term_n with a right-to-left orientation in the base ident.

3. Hint Rewrite term_1 ... term_n using tactic : ident
   When the rewriting rules term_1 ... term_n in ident will be used, the tactic tactic will be applied to the generated subgoals, the main subgoal excluded.

4. Print Rewrite HintDb ident
   This command displays all rewrite hints contained in ident.
8.13 Controlling automation

8.13.5 Hints and sections

Hints provided by the Hint commands are erased when closing a section. Conversely, all hints of a module A that are not defined inside a section (and not defined with option Local) become available when the module A is imported (using e.g. Require Import A.).

8.13.6 Setting implicit automation tactics

Proof with tactic.

This command may be used to start a proof. It defines a default tactic to be used each time a tactic command tactic1 is ended by “...”. In this case the tactic command typed by the user is equivalent to tactic1:tactic.

See also: Proof. in Section 7.1.5.

Declare Implicit Tactic tactic.

This command declares a tactic to be used to solve implicit arguments that Coq does not know how to solve by unification. It is used every time the term argument of a tactic has one of its holes not fully resolved.

Here is an example:

quo is assumed
Coq < Notation "x // y" := (quo x y _) (at level 40).
Coq <
Coq < Declare Implicit Tactic assumption.
Coq < Goal forall n m, m<>0 -> \{ q:nat & \{ r | q * m + r = n \} \}.
1 subgoal

1 subgoal
forall n m : nat, m <> 0 -> \{ q : nat & \{ r : nat | q * m + r = n \}\}
Coq < intros.
1 subgoal

n : nat
m : nat
H : m <> 0

1 subgoal

\{ q : nat & \{ r : nat | q * m + r = n \}\}
Coq < exists (n // m).
1 subgoal

n : nat
m : nat
H : m <> 0

1 subgoal

\{ r : nat | n // m * m + r = n \}

The tactic exists (n // m) did not fail. The hole was solved by assumption so that it behaved as exists (quo n m H).
8.14 Generation of induction principles with Scheme

The Scheme command is a high-level tool for generating automatically (possibly mutual) induction principles for given types and sorts. Its syntax follows the schema:

```
Scheme ident := Induction for ident' Sort sort
with
...with ident_m := Induction for ident'_m Sort sort_m
```

where ident' ... ident'_m are different inductive type identifiers belonging to the same package of mutual inductive definitions. This command generates ident...ident_m to be mutually recursive definitions. Each term ident_i proves a general principle of mutual induction for objects in type term_i.

Variants:

1. Scheme ident := Minimality for ident' Sort sort
with
...with ident_m := Minimality for ident'_m Sort sort_m

Same as before but defines a non-dependent elimination principle more natural in case of inductively defined relations.

2. Scheme Equality for ident

Tries to generate a boolean equality and a proof of the decidability of the usual equality.

3. Scheme Induction for ident Sort sort
with
...with Induction for ident_m Sort sort_m

If you do not provide the name of the schemes, they will be automatically computed from the sorts involved (works also with Minimality).

See also: Section 10.3

8.14.1 Automatic declaration of schemes

It is possible to deactivate the automatic declaration of the induction principles when defining a new inductive type with the Unset Elimination Schemes command. It may be reactivated at any time with Set Elimination Schemes.

You can also activate the automatic declaration of those boolean equalities (see the second variant of Scheme) with the Set Equality Scheme command. However you have to be careful with this option since COQ may now reject well-defined inductive types because it cannot compute a boolean equality for them.
8.14.2 Combined Scheme

The Combined Scheme command is a tool for combining induction principles generated by the Scheme command. Its syntax follows the schema:

\[
\text{Combined Scheme } \text{ident}_0 \text{ from } \text{ident}_1, \ldots, \text{ident}_n
\]

\text{ident}_1 \ldots \text{ident}_n \text{ are different inductive principles that must belong to the same package of mutual inductive principle definitions. This command generates } \text{ident}_0 \text{ to be the conjunction of the principles: it is built from the common premises of the principles and concluded by the conjunction of their conclusions.}

\text{See also: Section 10.3.1}

8.15 Generation of induction principles with Functional Scheme

The Functional Scheme command is a high-level experimental tool for generating automatically induction principles corresponding to (possibly mutually recursive) functions. Its syntax follows the schema:

\[
\text{Functional Scheme } \text{ident}_1 := \text{Induction for } \text{ident'}_1 \text{ Sort sort}_1 \\
\text{with} \\
\ldots \\
\text{with } \text{ident}_m := \text{Induction for } \text{ident'}_m \text{ Sort sort}_m
\]

\text{where } \text{ident'}_1 \ldots \text{ident'}_m \text{ are different mutually defined function names (they must be in the same order as when they were defined). This command generates the induction principles } \text{ident}_1 \ldots \text{ident}_m, \text{ following the recursive structure and case analyses of the functions } \text{ident'}_1 \ldots \text{ident'}_m.

\text{Functional Scheme} \text{ There is a difference between obtaining an induction scheme by using Functional Scheme on a function defined by Function or not. Indeed Function generally produces smaller principles, closer to the definition written by the user.}

\text{See also: Section 10.4}

8.16 Simple tactic macros

A simple example has more value than a long explanation:

\begin{verbatim}
Coq < Ltac Solve := simpl; intros; auto. 
Solve is defined
Coq < Ltac ElimBoolRewrite b H1 H2 :=
Coq < elim b; [ intros; rewrite H1; eauto | intros; rewrite H2; eauto ].
ElimBoolRewrite is defined
\end{verbatim}

The tactics macros are synchronous with the COQ section mechanism: a tactic definition is deleted from the current environment when you close the section (see also 2.4) where it was defined. If you want that a tactic macro defined in a module is usable in the modules that require it, you should put it outside of any section.

Chapter 9 gives examples of more complex user-defined tactics.
Chapter 9

The tactic language

This chapter gives a compact documentation of Ltac, the tactic language available in Coq. We start by giving the syntax, and next, we present the informal semantics. If you want to know more regarding this language and especially about its foundations, you can refer to [41]. Chapter 10 is devoted to giving examples of use of this language on small but also with non-trivial problems.

9.1 Syntax

The syntax of the tactic language is given Figures 9.1 and 9.2. See Chapter 1 for a description of the BNF metasyntax used in these grammar rules. Various already defined entries will be used in this chapter: entries natural, integer, ident, qualid, term, cpattern and atomic_tactic represent respectively the natural and integer numbers, the authorized identifiers and qualified names, Coq’s terms and patterns and all the atomic tactics described in Chapter 8. The syntax of cpattern is the same as that of terms, but it is extended with pattern matching metavariables. In cpattern, a pattern-matching metavariable is represented with the syntax ?id where id is an ident. The notation _ can also be used to denote metavariable whose instance is irrelevant. In the notation ?id, the identifier allows us to keep instantiations and to make constraints whereas _ shows that we are not interested in what will be matched. On the right hand side of pattern-matching clauses, the named metavariable are used without the question mark prefix. There is also a special notation for second-order pattern-matching problems: in an applicative pattern of the form @?id id₁ ...idₙ, the variable id matches any complex expression with (possible) dependencies in the variables id₁ ...idₙ and returns a functional term of the form fun id₁ ...idₙ => term.

The main entry of the grammar is expr. This language is used in proof mode but it can also be used in toplevel definitions as shown in Figure 9.3.

Remarks:

1. The infix tacticals “... || ...” and “... ; ...” are associative.

2. In tacarg, there is an overlap between qualid as a direct tactic argument and qualid as a particular case of term. The resolution is done by first looking for a reference of the tactic language and if it fails, for a reference to a term. To force the resolution as a reference of the tactic language, use the form ltac : qualid. To force the resolution as a reference to a term, use the syntax (qualid).

3. As shown by the figure, tactical || binds more than the prefix tacticals try, repeat, do, info and abstract which themselves bind more than the postfix tactical “... ; [ ... ]” which binds more than “... ; ...”.

Coq Reference Manual, V8.2pl2, June 29, 2010
For instance

\[
\text{try repeat } \text{tactic}_1 \mid | \text{tactic}_2; \text{tactic}_3; [\text{tactic}_{31} | \ldots | \text{tactic}_{3n}] ; \text{tactic}_4.
\]

is understood as

\[
(\text{try (repeat (tactic}_1 \mid | \text{tactic}_2))));
(\text{((tactic}_3; [\text{tactic}_{31} | \ldots | \text{tactic}_{3n}] ) ; \text{tactic}_4).
\]

### 9.2 Semantics

Tactic expressions can only be applied in the context of a goal. The evaluation yields either a term, an integer or a tactic. Intermediary results can be terms or integers but the final result must be a tactic which is then applied to the current goal.

There is a special case for match goal expressions of which the clauses evaluate to tactics. Such expressions can only be used as end result of a tactic expression (never as argument of a non recursive local definition or of an application).

The rest of this section explains the semantics of every construction of Ltac.

#### Sequence

A sequence is an expression of the following form:

\[
expr_1 ; \ expr_2
\]

The expressions \(expr_1\) and \(expr_2\) are evaluated to \(v_1\) and \(v_2\) which have to be tactic values. The tactic \(v_1\) is then applied and \(v_2\) is applied to every subgoal generated by the application of \(v_1\). Sequence is left-associative.

#### General sequence

A general sequence has the following form:

\[
expr_0 ; [ expr_1 | \ldots | expr_n ]
\]

The expressions \(expr_i\) are evaluated to \(v_i\), for \(i = 0, \ldots, n\) and all have to be tactics. The tactic \(v_0\) is applied and \(v_i\) is applied to the \(i\)-th generated subgoal by the application of \(v_0\), for \(i = 1, \ldots, n\). It fails if the application of \(v_0\) does not generate exactly \(n\) subgoals.

**Variants:**

1. If no tactic is given for the \(i\)-th generated subgoal, it behaves as if the tactic \text{idtac} were given. For instance, \text{split} ; [ \mid \text{auto} \] is a shortcut for \text{split} ; [ \text{idtac} | \text{auto} ].

2. \(expr_0 ; [ expr_1 | \ldots | expr_i | \ldots | expr_{i+1+j} | \ldots | expr_n ]\)

   In this variant, \text{idtac} is used for the subgoals numbered from \(i + 1\) to \(i + j\) (assuming \(n\) is the number of subgoals).

   Note that \ldots is part of the syntax, while \ldots is the meta-symbol used to describe a list of \text{expr} of arbitrary length.

3. \(expr_0 ; [ expr_1 | \ldots | expr_i | expr \ldots | expr_{i+1+j} | \ldots | expr_n ]\)

   In this variant, \text{expr} is used instead of \text{idtac} for the subgoals numbered from \(i + 1\) to \(i + j\).
Figure 9.1: Syntax of the tactic language
The tactic language

\[
tacarg ::= \text{qualid} \mid () \mid \text{l tac} : \text{atom} \mid \text{term}
\]

\[
\text{let_clause ::= ident \{name \ldots name\} := expr}
\]

\[
\text{context_rule ::= context_hyp, \ldots, context_hyp \vdash \text{-cpattern} \Rightarrow expr} \mid \text{\vdash \text{-cpattern} \Rightarrow expr} \mid _ \Rightarrow expr
\]

\[
\text{context_hyp ::= name : cpattern} \mid \text{name := cpattern \{ cpattern \}}
\]

\[
\text{match_rule ::= cpattern \Rightarrow expr} \mid \text{context \{ident\} \{ cpattern \} \Rightarrow expr} \mid _ \Rightarrow expr
\]

For loop

There is a for loop that repeats a tactic \text{num} times:

\[
\text{do num expr}
\]

\text{expr} is evaluated to \text{v}. \text{v} must be a tactic value. \text{v} is applied \text{num} times. Supposing \text{num} > 1, after the first application of \text{v}, \text{v} is applied, at least once, to the generated subgoals and so on. It fails if the application of \text{v} fails before the \text{num} applications have been completed.

Repeat loop

We have a repeat loop with:

\[
\text{repeat expr}
\]

\text{expr} is evaluated to \text{v}. If \text{v} denotes a tactic, this tactic is applied to the goal. If the application fails, the tactic is applied recursively to all the generated subgoals until it eventually fails. The recursion stops in a subgoal when the tactic has failed. The tactic \text{repeat expr} itself never fails.

Error catching

We can catch the tactic errors with:

\[
\text{Coq Reference Manual, V8.2p12, June 29, 2010}
\]
try expr

expr is evaluated to \( v \). \( v \) must be a tactic value. \( v \) is applied. If the application of \( v \) fails, it catches the error and leaves the goal unchanged. If the level of the exception is positive, then the exception is re-raised with its level decremented.

**Detecting progress**

We can check if a tactic made progress with:

```coq
progress expr
```

expr is evaluated to \( v \). \( v \) must be a tactic value. \( v \) is applied. If the application of \( v \) produced one subgoal equal to the initial goal (up to syntactical equality), then an error of level 0 is raised.

**Error message:** Failed to progress

**Branching**

We can easily branch with the following structure:

```coq
expr_1 \parallel expr_2
```

\( expr_1 \) and \( expr_2 \) are evaluated to \( v_1 \) and \( v_2 \). \( v_1 \) and \( v_2 \) must be tactic values. \( v_1 \) is applied and if it fails to progress then \( v_2 \) is applied. Branching is left-associative.

**First tactic to work**

We may consider the first tactic to work (i.e. which does not fail) among a panel of tactics:

```coq
first [ expr_1 \mid \ldots \mid expr_n ]
```

\( expr_i \) are evaluated to \( v_i \) and \( v_i \) must be tactic values, for \( i = 1, \ldots, n \). Supposing \( n > 1 \), it applies \( v_1 \), if it works, it stops else it tries to apply \( v_2 \) and so on. It fails when there is no applicable tactic.

**Error message:** No applicable tactic

**Solving**

We may consider the first to solve (i.e. which generates no subgoal) among a panel of tactics:

```coq
solve [ expr_1 \mid \ldots \mid expr_n ]
```

\( expr_i \) are evaluated to \( v_i \) and \( v_i \) must be tactic values, for \( i = 1, \ldots, n \). Supposing \( n > 1 \), it applies \( v_1 \), if it solves, it stops else it tries to apply \( v_2 \) and so on. It fails if there is no solving tactic.

**Error message:** Cannot solve the goal

**Identity**

The constant \texttt{idtac} is the identity tactic: it leaves any goal unchanged but it appears in the proof script.

**Variant:** \texttt{idtac message_token \ldots message_token}

This prints the given tokens. Strings and integers are printed literally. If a term is given, it is printed, its variables being interpreted in the current environment. In particular, if a variable is given, its value is printed.
Failing

The tactic fail is the always-failing tactic: it does not solve any goal. It is useful for defining other tacticals since it can be caught by try or match goal.

Variants:

1. fail $n$
   The number $n$ is the failure level. If no level is specified, it defaults to 0. The level is used by try and match goal. If 0, it makes match goal considering the next clause (backtracking). If non zero, the current match goal block or try command is aborted and the level is decremented.

2. fail message_token ... message_token
   The given tokens are used for printing the failure message.

3. fail $n$ message_token ... message_token
   This is a combination of the previous variants.

Error message: Tactic Failure message (level $n$).

Local definitions

Local definitions can be done as follows:

```
let ident$_1$: expr$_1$
with ident$_2$: expr$_2$
... 
with ident$_n$: expr$_n$ in 
expr
```

each expr$_i$ is evaluated to $v_i$, then, expr is evaluated by substituting $v_i$ to each occurrence of ident$_i$, for $i = 1, ..., n$. There is no dependencies between the expr$_i$ and the ident$_i$.

Local definitions can be recursive by using let rec instead of let. In this latter case, the definitions are evaluated lazily so that the rec keyword can be used also in non recursive cases so as to avoid the eager evaluation of local definitions.

Application

An application is an expression of the following form:

```
qualid tacarg$_1$ ... tacarg$_n$
```

The reference qualid must be bound to some defined tactic definition expecting at least $n$ arguments. The expressions expr$_i$ are evaluated to $v_i$, for $i = 1, ..., n$.

Function construction

A parameterized tactic can be built anonymously (without resorting to local definitions) with:

```
fun ident$_1$ ... ident$_n$ => expr
```

Indeed, local definitions of functions are a syntactic sugar for binding a fun tactic to an identifier.
Pattern matching on terms

We can carry out pattern matching on terms with:

```
match expr with
    cpattern₁ => expr₁
  | cpattern₂ => expr₂
  ...
  | cpatternₙ => exprₙ
  | _ => exprₙ₊₁
end
```

The expression `expr` is evaluated and should yield a term which is matched against `cpattern₁`. The matching is non-linear: if a metavariable occurs more than once, it should match the same expression every time. It is first-order except on the variables of the form `@?id` that occur in head position of an application. For these variables, the matching is second-order and returns a functional term.

If the matching with `cpattern₁` succeeds, then `expr₁` is evaluated into some value by substituting the pattern matching instantiations to the metavariables. If `expr₁` evaluates to a tactic and the `match` expression is in position to be applied to a goal (e.g. it is not bound to a variable by a `let in`), then this tactic is applied. If the tactic succeeds, the list of resulting subgoals is the result of the `match` expression. If `expr₁` does not evaluate to a tactic or if the `match` expression is not in position to be applied to a goal, then the result of the evaluation of `expr₁` is the result of the `match` expression.

If the matching with `cpattern₁` fails, or if it succeeds but the evaluation of `expr₁` fails, or if the evaluation of `expr₁` succeeds but returns a tactic in execution position whose execution fails, then `cpattern₂` is used and so on. The pattern `_` matches any term and shunts all remaining patterns if any. If all clauses fail (in particular, there is no pattern `_`) then a no-matching-clause error is raised.

Error messages:

1. No matching clauses for `match`
   No pattern can be used and, in particular, there is no `_` pattern.

2. Argument of `match` does not evaluate to a term
   This happens when `expr` does not denote a term.

Variants:

1. There is a special form of patterns to match a subterm against the pattern:

   ```
   context ident [ cpattern ]
   ```

   It matches any term which one subterm matches `cpattern`. If there is a match, the optional `ident` is assign the “matched context”, that is the initial term where the matched subterm is replaced by a hole. The definition of `context` in expressions below will show how to use such term contexts.

   If the evaluation of the right-hand-side of a valid match fails, the next matching subterm is tried. If no further subterm matches, the next clause is tried. Matching subterms are considered top-bottom and from left to right (with respect to the raw printing obtained by setting option `Printing All`, see Section 2.9).
Coq < Ltac f x :=
Coq < match x with
Coq < context f [S ?X] =>
Coq < idtac X; (* To display the evaluation order *)
Coq < assert (p := refl_equal 1 : X=1); (* To filter the case X=1 *)
Coq < let x:= context f[O] in assert (x=O) (* To observe the context *)
Coq < end.

f is defined

Coq < Goal True.
1 subgoal

============================
True

Coq < f (3+4).
2
1
2 subgoals

p : 1 = 1

==================================
1 + 4 = 0
subgoal 2 is:
True

2. Using lazymatch instead of match has an effect if the right-hand-side of a clause returns a tactic. With match, the tactic is applied to the current goal (and the next clause is tried if it fails). With lazymatch, the tactic is directly returned as the result of the whole lazymatch block without being first tried to be applied to the goal. Typically, if the lazymatch block is bound to some variable \( x \) in a let in, then tactic expression gets bound to the variable \( x \).

Pattern matching on goals

We can make pattern matching on goals using the following expression:

\[
\text{match goal with } \\
| \text{hyp}_{1,1}, \ldots, \text{hyp}_{1,m_1} | \neg \text{pattern}_1 \Rightarrow \text{expr}_1 \\
| \text{hyp}_{2,1}, \ldots, \text{hyp}_{2,m_2} | \neg \text{pattern}_2 \Rightarrow \text{expr}_2 \\
\cdots \\
| \text{hyp}_{n,1}, \ldots, \text{hyp}_{n,m_n} | \neg \text{pattern}_n \Rightarrow \text{expr}_n \\
| \_ \Rightarrow \text{expr}_{n+1} \\
\text{end }
\]

If each hypothesis pattern \( \text{hyp}_{i,i} \), with \( i = 1, \ldots, m_1 \) is matched (non-linear first-order unification) by an hypothesis of the goal and if \( \text{pattern}_i \) is matched by the conclusion of the goal, then \( \text{expr}_i \) is evaluated to \( v_i \) by substituting the pattern matching to the metavariables and the real hypothesis names bound to the possible hypothesis names occurring in the hypothesis patterns. If \( v_i \) is a tactic value, then it is applied to the goal. If this application fails, then another combination of hypotheses is tried with the same proof context pattern. If there is no other combination of hypotheses then the second proof context pattern is tried and so on. If the next to last proof context pattern fails then \( \text{expr}_{n+1} \) is evaluated to \( v_{n+1} \).
and \( v_{n+1} \) is applied. Note also that matching against subterms (using the context `ident [ cpattern ]`) is available and may itself induce extra backtracking.

**Error message:** No matching clauses for match goal

No clause succeeds, i.e. all matching patterns, if any, fail at the application of the right-hand-side.

It is important to know that each hypothesis of the goal can be matched by at most one hypothesis pattern. The order of matching is the following: hypothesis patterns are examined from the right to the left (i.e. \( hyp_{i,m} \), before \( hyp_{i,1} \)). For each hypothesis pattern, the goal hypothesis are matched in order (fresher hypothesis first), but it possible to reverse this order (older first) with the `match reverse goal with variant`.

**Variant:** Using `lazymatch` instead of `match` has an effect if the right-hand-side of a clause returns a tactic. With `match`, the tactic is applied to the current goal (and the next clause is tried if it fails). With `lazymatch`, the tactic is directly returned as the result of the whole `lazymatch` block without being first tried to be applied to the goal. Typically, if the `lazymatch` block is bound to some variable \( x \) in a `let in`, then tactic expression gets bound to the variable \( x \).

```coq
Ltac test_lazy :=
  lazymatch goal with
  | _ => idtac "here"; fail
  | _ => idtac "wasn't lazy"; trivial
  end.
test_lazy is defined

Ltac test_eager :=
  match goal with
  | _ => idtac "here"; fail
  | _ => idtac "wasn't lazy"; trivial
  end.
test_eager is defined

Goal True.
1 subgoal

\[ \text{-----------------------------} \]
\[ \text{True} \]

test_lazy || idtac "was lazy".
here
was lazy
1 subgoal

\[ \text{-----------------------------} \]
\[ \text{True} \]

test_eager || idtac "was lazy".
here
wasn't lazy
Proof completed.
```

**Filling a term context**

The following expression is not a tactic in the sense that it does not produce subgoals but generates a term to be used in tactic expressions:
context ident [ expr ]

Ident must denote a context variable bound by a context pattern of a match expression. This expression evaluates replaces the hole of the value of ident by the value of expr.

Error message: not a context variable

Generating fresh hypothesis names

Tactics sometimes have to generate new names for hypothesis. Letting the system decide a name with the intro tactic is not so good since it is very awkward to retrieve the name the system gave. The following expression returns an identifier:

fresh component ... component

It evaluates to an identifier unbound in the goal. This fresh identifier is obtained by concatenating the value of the component’s (each of them is, either an ident which has to refer to a name, or directly a name denoted by a string). If the resulting name is already used, it is padded with a number so that it becomes fresh. If no component is given, the name is a fresh derivative of the name H.

Computing in a constr

Evaluation of a term can be performed with:

eval redexpr in term

where redexpr is a reduction tactic among \texttt{red, hnf, compute, simpl, cbv, lazy, unfold, fold, pattern}.

Type-checking a term

The following returns the type of term:

type of term

Accessing tactic decomposition

Tactical “info expr” is not really a tactical. For elementary tactics, this is equivalent to expr. For complex tactic like \texttt{auto}, it displays the operations performed by the tactic.

Proving a subgoal as a separate lemma

From the outside “abstract expr” is the same as solve expr. Internally it saves an auxiliary lemma called \texttt{ident_subproof} where \texttt{ident} is the name of the current goal and \texttt{n} is chosen so that this is a fresh name.

This tactical is useful with tactics such as \texttt{omega} or \texttt{discriminate} that generate huge proof terms. With that tool the user can avoid the explosion at time of the \texttt{Save} command without having to cut manually the proof in smaller lemmas.

Variants:

1. abstract expr using ident.
   
   Give explicitly the name of the auxiliary lemma.

Error message: Proof is not complete
9.3 Tactic toplevel definitions

Calling an external tactic

The tactic \texttt{external} allows to run an executable outside the COQ executable. The communication is done via an XML encoding of constructions. The syntax of the command is

\begin{verbatim}
external "command" "request" tacarg ... tacarg
\end{verbatim}

The string \texttt{command}, to be interpreted in the default execution path of the operating system, is the name of the external command. The string \texttt{request} is the name of a request to be sent to the external command. Finally the list of tactic arguments have to evaluate to terms. An XML tree of the following form is sent to the standard input of the external command.

\begin{verbatim}
<REQUEST req="request">
the XML tree of the first argument
...
the XML tree of the last argument
</REQUEST>
\end{verbatim}

Conversely, the external command must send on its standard output an XML tree of the following forms:

\begin{verbatim}
<TERM>
the XML tree of a term
</TERM>
\end{verbatim}
or

\begin{verbatim}
<CALL uri="ltacQualifiedIdent">
the XML tree of a first argument
...
the XML tree of a last argument
</CALL>
\end{verbatim}

where \texttt{ltacQualifiedIdent} is the name of a defined \( \mathcal{L}_{\text{tac}} \) function and each subsequent XML tree is recursively a \texttt{CALL} or a \texttt{TERM} node.

The Document Type Definition (DTD) for terms of the Calculus of Inductive Constructions is the one developed as part of the MoWGLI European project. It can be found in the file \texttt{dev/doc/cic.dtd} of the COQ source archive.

An example of parser for this DTD, written in the Objective Caml - Camlp4 language, can be found in the file \texttt{parsing/g_xml.ml} of the COQ source archive.

9.3 Tactic toplevel definitions

9.3.1 Defining \( \mathcal{L}_{\text{tac}} \) functions

Basically, \( \mathcal{L}_{\text{tac}} \) toplevel definitions are made as follows:

\begin{verbatim}
Ltac ident ident_1 ... ident_n := expr
\end{verbatim}

This defines a new \( \mathcal{L}_{\text{tac}} \) function that can be used in any tactic script or new \( \mathcal{L}_{\text{tac}} \) toplevel definition.

Remark: The preceding definition can equivalently be written:

\begin{verbatim}
Ltac ident := fun ident_1 ... ident_n => expr
\end{verbatim}
Recursive and mutual recursive function definitions are also possible with the syntax:

\[ \text{Ltac } \text{ident}_1 \text{ ident}_{1,1} \ldots \text{ ident}_{1,m_1} := \text{expr}_1 \]
with \[ \text{ident}_2 \text{ ident}_{2,1} \ldots \text{ ident}_{2,m_2} := \text{expr}_2 \]
\[ \ldots \]
with \[ \text{ident}_n \text{ ident}_{n,1} \ldots \text{ ident}_{n,m_n} := \text{expr}_n \]

It is also possible to redefine an existing user-defined tactic using the syntax:

\[ \text{Ltac } \text{qualid } \text{ident}_1 \ldots \text{ ident}_n := \text{expr} \]

A previous definition of \text{qualid} must exist in the environment. The new definition will always be used instead of the old one and it goes across module boundaries.

### 9.3.2 Printing \text{Ltac} tactics

Defined \text{Ltac} functions can be displayed using the command

\[ \text{Print Ltac qualid} . \]

### 9.4 Debugging \text{Ltac} tactics

The \text{Ltac} interpreter comes with a step-by-step debugger. The debugger can be activated using the command

\[ \text{Set Ltac Debug} . \]

and deactivated using the command

\[ \text{Unset Ltac Debug} . \]

To know if the debugger is on, use the command \text{Test Ltac Debug}. When the debugger is activated, it stops at every step of the evaluation of the current \text{Ltac} expression and it prints information on what it is doing. The debugger stops, prompting for a command which can be one of the following:

- simple newline: go to the next step
- h: get help
- x: exit current evaluation
- s: continue current evaluation without stopping
- \text{\texttt{\textbackslash r\text{n}}} \text{\textbackslash n}: advance \text{n} steps further
Chapter 10

Detailed examples of tactics

This chapter presents detailed examples of certain tactics, to illustrate their behavior.

10.1 refine

This tactic applies to any goal. It behaves like exact with a big difference: the user can leave some
holes (denoted by _ or (_:type)) in the term. refine will generate as many subgoals as they are
holes in the term. The type of holes must be either synthesized by the system or declared by an explicit
cast like (\_:nat->Prop). This low-level tactic can be useful to advanced users.

Example:

Coq < Inductive Option : Set :=
Coq < | Fail : Option
Coq < | Ok : bool -> Option.
Coq < Definition get : forall x:Option, x <> Fail -> bool.
1 subgoal

============
forall x : Option, x <> Fail -> bool
Coq < refine
Coq < (fun x:Option =>
Coq < match x return x <> Fail -> bool with
Coq < | Fail => _
Coq < | Ok b => fun _ => b
Coq < end).
1 subgoal

x : Option

============= Fail <> Fail -> bool
Coq < intros; absurd (Fail = Fail); trivial.
Proof completed.
Coq < Defined.
10.2 eapply

**Example:** Assume we have a relation on \( \text{nat} \) which is transitive:

Coq < Variable \( R : \text{nat} \rightarrow \text{nat} \rightarrow \text{Prop} \).
Coq < Hypothesis Rtrans : \( \forall x y z: \text{nat}, R x y \rightarrow R y z \rightarrow R x z \).
Coq < Variables n m p : \text{nat}.
Coq < Hypothesis Rnm : R n m.
Coq < Hypothesis Rmp : R m p.

Consider the goal \( (R n p) \) provable using the transitivity of \( R \):

Coq < Goal R n p.

The direct application of \( \text{Rtrans} \) with `apply` fails because no value for \( y \) in \( \text{Rtrans} \) is found by `apply`:

Coq < apply Rtrans.

Error: Unable to find an instance for the variable \( y \).

A solution is to rather apply \( (\text{Rtrans} n m p) \).

Coq < apply (Rtrans n m p).

More elegantly, apply \( \text{Rtrans} \) with \( (y:=m) \) allows to only mention the unknown \( m \):

Coq < apply Rtrans with (y := m).

Another solution is to mention the proof of \( (R x y) \) in \( \text{Rtrans} \...
... or the proof of $(R \ y \ z)$:

Coq <
Coq <  apply Rtrans with (2 := Rmp).
1 subgoal

============================
R n m

On the opposite, one can use eapply which postpone the problem of finding m. Then one can apply the hypotheses $R_{nm}$ and $R_{mp}$. This instantiates the existential variable and completes the proof.

Coq < eapply Rtrans.
2 subgoals

============================
R n ?9
subgoal 2 is:
R ?9 p
Coq < apply Rnm.
1 subgoal

============================
R m p
Coq < apply Rmp.
Proof completed.

10.3 Scheme

Example 1: Induction scheme for tree and forest

The definition of principle of mutual induction for tree and forest over the sort Set is defined by the command:

Coq < Inductive tree : Set :=
Coq <   node : A -> forest -> tree
Coq < with forest : Set :=
Coq <   | leaf : B -> forest
Coq <   | cons : tree -> forest -> forest.
Coq <
Coq < Scheme tree_forest_rec := Induction for tree Sort Set
Coq <   with forest_tree_rec := Induction for forest Sort Set.

You may now look at the type of $\text{tree}_{\text{forest}}_{\text{rec}}$:

Coq < Check tree_forest_rec.

$\text{tree}_{\text{forest}}_{\text{rec}}$

: forall (P : tree -> Set) (P0 : forest -> Set),
(forall (a : A) (f : forest), P0 f -> P (node a f)) ->
(forall b : B, P0 (leaf b)) ->
(forall t : tree, P t -> forall f1 : forest, P0 f1 -> P0 (cons t f1)) ->
forall t : tree, P t
This principle involves two different predicates for trees and forests; it also has three premises each one corresponding to a constructor of one of the inductive definitions.

The principle forest_tree_rec shares exactly the same premises, only the conclusion now refers to the property of forests.

Example 2: Predicates odd and even on naturals

Let odd and even be inductively defined as:

```coq
Inductive odd : nat -> Prop :=
  oddS : forall n:nat, even n -> odd (S n)
with even : nat -> Prop :=
  | evenO : even 0
  | evenS : forall n:nat, odd n -> even (S n).
```

The following command generates a powerful elimination principle:

```coq
Scheme odd_even := Minimality for odd Sort Prop
with even_odd := Minimality for even Sort Prop.
odd_even, even_odd are recursively defined
```

The type of odd_even for instance will be:

```coq
Check odd_even.
odd_even
  : forall P P0 : nat -> Prop,
    (forall n : nat, even n -> P n) ->
    P0 0 ->
    (forall n : nat, odd n -> P n -> P0 (S n)) ->
    forall n : nat, odd n -> P n
```

The type of even_odd shares the same premises but the conclusion is \((n:nat)\ (\text{even } n) \rightarrow (Q n)\).

10.3.1 Combined Scheme

We can define the induction principles for trees and forests using:

```coq
Scheme tree_forest_ind := Induction for tree Sort Prop
with forest_tree_ind := Induction for forest Sort Prop.
tree_forest_ind, forest_tree_ind are recursively defined
```

Then we can build the combined induction principle which gives the conjunction of the conclusions of each individual principle:
10.4 Functional Scheme and functional induction

Coq < Combined Scheme tree_forest_mutind from tree_forest_ind, forest_tree_ind. 

The type of tree_forest_mutrec will be:

The definition of a principle of induction corresponding to the recursive structure of div2 is defined by the command:

You may now look at the type of div2_ind:

We can now prove the following lemma using this principle:

Example 1: Induction scheme for div2

We define the function div2 as follows:

Coq < Require Import Arith.

The definition of div2_ind is defined

div2_equation is defined

div2_ind is defined

You may now look at the type of div2_ind:

We can now prove the following lemma using this principle:

Coq < Lemma div2_le' : forall n:nat, div2 n <= n.

Coq < intro n.
Coq < pattern n , (div2 n).
Coq < apply div2_ind; intros.  
3 subgoals  
  n : nat  
  n0 : nat  
  e : n0 = 0  
  =============================  
  0 <= 0  
subgoal 2 is:  
  0 <= 1  
subgoal 3 is:  
  S (div2 n') <= S (S n')  
Coq < auto with arith.  
Coq < auto with arith.  
Coq < simpl; auto with arith.  
Coq < Qed.  

We can use directly the functional induction (8.7.7) tactic instead of the pattern/apply trick:  
Coq < Reset div2_le'.  
Coq < Lemma div2_le : forall n:nat, div2 n <= n.  
Coq < intro n.  
Coq < functional induction (div2 n).  
3 subgoals  
  =============================  
  0 <= 0  
subgoal 2 is:  
  0 <= 1  
subgoal 3 is:  
  S (div2 n') <= S (S n')  
Coq < auto with arith.  
Coq < auto with arith.  
Coq < auto with arith.  
Coq < Qed.  

Remark: There is a difference between obtaining an induction scheme for a function by using Function (see Section 2.3) and by using Functional Scheme after a normal definition using Fixpoint or Definition. See 2.3 for details.  

Example 2: Induction scheme for tree_size  
We define trees by the following mutual inductive type:  
Coq < Variable A : Set.  
Coq < Inductive tree : Set :=  
Coq <  node : A -> forest -> tree  
Coq < with forest : Set :=  
Coq <  | empty : forest  
Coq <  | cons : tree -> forest -> forest.
We define the function `tree_size` that computes the size of a tree or a forest. Note that we use `Function` which generally produces better principles.

```coq
Function tree_size (t:tree) : nat :=
  match t with
  | node A f => S (forest_size f)
  end
with
Function forest_size (f:forest) : nat :=
  match f with
  | empty => 0
  | cons t f' => (tree_size t + forest_size f')
end.
```

**Remark:** `Function` generates itself non mutual induction principles `tree_size_ind` and `forest_size_ind`:

```coq
Check tree_size_ind.
tree_size_ind
  : forall P : tree -> nat -> Prop,
  (forall (t : tree) (A : A) (f : forest),
   t = node A f -> P (node A f) (S (forest_size f))) ->
  forall t : tree, P t (tree_size t)
```

The definition of mutual induction principles following the recursive structure of `tree_size` and `forest_size` is defined by the command:

```coq
Functional Scheme tree_size_ind2 := Induction for tree_size Sort Prop
  with forest_size_ind2 := Induction for forest_size Sort Prop.
```

You may now look at the type of `tree_size_ind2`:

```coq
Check tree_size_ind2.
tree_size_ind2
  : forall (P : tree -> nat -> Prop) (P0 : forest -> nat -> Prop),
  (forall (t : tree) (A : A) (f : forest),
   t = node A f ->
   P0 f (forest_size f) -> P (node A f) (S (forest_size f))) ->
  (forall f0 : forest, f0 = empty -> P0 empty 0) ->
  (forall (f1 : forest) (t : tree) (f' : forest),
   f1 = cons t f' ->
   P t (tree_size t) ->
   P0 f' (forest_size f') ->
   P0 (cons t f') (tree_size t + forest_size f')) ->
  forall t : tree, P t (tree_size t)
```

### 10.5 inversion

**Generalities about inversion**

When working with (co)inductive predicates, we are very often faced to some of these situations:

- we have an inconsistent instance of an inductive predicate in the local context of hypotheses. Thus, the current goal can be trivially proved by absurdity.
• we have a hypothesis that is an instance of an inductive predicate, and the instance has some variables whose constraints we would like to derive.

The inversion tactics are very useful to simplify the work in these cases. Inversion tools can be classified in three groups:

1. tactics for inverting an instance without stocking the inversion lemma in the context; this includes the tactics (dependent) inversion and (dependent) inversion_clear.

2. commands for generating and stocking in the context the inversion lemma corresponding to an instance; this includes Derive (Dependent) Inversion and Derive (Dependent) Inversion_clear.

3. tactics for inverting an instance using an already defined inversion lemma; this includes the tactic inversion ...using.

As inversion proofs may be large in size, we recommend the user to stock the lemmas whenever the same instance needs to be inverted several times.

**Example 1: Non-dependent inversion**

Let’s consider the relation \(\text{Le}\) over natural numbers and the following variables:

\[
\begin{align*}
\text{Coq} &< \text{Inductive } \text{Le} : \text{nat} \rightarrow \text{nat} \rightarrow \text{Set} := \\
&\quad | \text{LeO} : \forall n : \text{nat}, \text{Le} 0 n \\
&\quad | \text{LeS} : \forall n m : \text{nat}, \text{Le} n m \rightarrow \text{Le} (S n) (S m). \\
\text{Coq} &< \text{Variable } P : \forall \text{nat} \rightarrow \text{nat} \rightarrow \text{Prop}. \\
\text{Coq} &< \text{Variable } Q : \forall \text{nat} \rightarrow \text{nat} \rightarrow \text{nat} \rightarrow \text{Prop}. 
\end{align*}
\]

For example, consider the goal:

\[
\begin{align*}
\text{Coq} &< \text{Show}. \\
&1 \text{ subgoal} \\
&n : \text{nat} \\
m : \text{nat} \\
H : \text{Le} (S n) m \\
&\text{============================} \\
&P n m 
\end{align*}
\]

To prove the goal we may need to reason by cases on \(H\) and to derive that \(m\) is necessarily of the form \((S m_0)\) for certain \(m_0\) and that \((\text{Le} n m_0)\). Deriving these conditions corresponds to prove that the only possible constructor of \((\text{Le} (S n) m)\) is \(\text{LeS}\) and that we can invert the \(\rightarrow\) in the type of \(\text{LeS}\). This inversion is possible because \(\text{Le}\) is the smallest set closed by the constructors \(\text{LeO}\) and \(\text{LeS}\).

\[
\begin{align*}
\text{Coq} &< \text{inversion\_clear } H. \\
&1 \text{ subgoal} \\
&n : \text{nat} \\
m : \text{nat} \\
m_0 : \text{nat} \\
H_0 : \text{Le} n m_0 \\
&\text{============================} \\
&P n (S m_0)
\end{align*}
\]
Note that $m$ has been substituted in the goal for $(S\ m0)$ and that the hypothesis $(Le\ n\ m0)$ has been added to the context.

Sometimes it is interesting to have the equality $m=(S\ m0)$ in the context to use it after. In that case we can use inversion that does not clear the equalities:

Coq < Undo.
Coq < inversion H.
1 subgoal

$n : nat$
$m : nat$
$H : Le\ (S\ n)\ m$
$m0 : nat$
$H0 : n0 = n$
$H2 : S\ m0 = m$

$\Rightarrow$-----------------------------
$P\ n\ (S\ m0)$

**Example 2: Dependent Inversion**

Let us consider the following goal:

Coq < Show.
1 subgoal

$n : nat$
$m : nat$
$H : Le\ (S\ n)\ m$

$\Rightarrow$-----------------------------
$Q\ (S\ n)\ m\ H$

As $H$ occurs in the goal, we may want to reason by cases on its structure and so, we would like inversion tactics to substitute $H$ by the corresponding term in constructor form. Neither Inversion nor Inversion_clear make such a substitution. To have such a behavior we use the dependent inversion tactics:

Coq < dependent inversion_clear H.
1 subgoal

$n : nat$
$m : nat$
$m0 : nat$
$l : Le\ n\ m0$

$\Rightarrow$-----------------------------
$Q\ (S\ n)\ (S\ m0)\ (LeS\ n\ m0\ l)$

Note that $H$ has been substituted by $(LeS\ n\ m0\ l)$ and $m$ by $(S\ m0)$.

**Example 3: using already defined inversion lemmas**

For example, to generate the inversion lemma for the instance $(Le\ (S\ n)\ m)$ and the sort Prop we do:
Coq < Derive Inversion_clear leminv with (forall n m:nat, Le (S n) m) Sort
Coq < Prop.

Coq < Check leminv.
leminv
  : forall (n m : nat) (P : nat -> nat -> Prop),
    (forall m0 : nat, Le n m0 -> P n (S m0)) -> Le (S n) m -> P n m

Then we can use the proven inversion lemma:

Coq < Show.
1 subgoal

n : nat
m : nat
H : Le (S n) m

Then we can use the proven inversion lemma:

Coq < inversion H using leminv.
1 subgoal

n : nat
m : nat
H : Le (S n) m

10.6 dependent induction

The tactics dependent induction and dependent destruction are another solution for inverting inductive predicate instances and potentially doing induction at the same time. It is based on the BasicElim tactic of Conor McBride which works by abstracting each argument of an inductive instance by a variable and constraining it by equalities afterwards. This way, the usual induction and destruct tactics can be applied to the abstracted instance and after simplification of the equalities we get the expected goals.

The abstracting tactic is called generalize_eqs and it takes as argument an hypothesis to generalize. It uses the JMeq datatype defined in Coq.Logic.JMeq, hence we need to require it before. For example, revisiting the first example of the inversion documentation above:

Coq < Require Import Coq.Logic.JMeq.

Coq < Goal forall n m:nat, Le (S n) m -> P n m.
Coq < intros n m H.
Coq < generalize_eqs H.
1 subgoal

n : nat
m : nat
gen_x : nat
The index $S \; n$ gets abstracted by a variable here, but a corresponding equality is added under the abstract instance so that no information is actually lost. The goal is now almost amenable to do induction or case analysis. One should indeed first move $n$ into the goal to strengthen it before doing induction, or $n$ will be fixed in the inductive hypotheses (this does not matter for case analysis). As a rule of thumb, all the variables that appear inside constructors in the indices of the hypothesis should be generalized. This is exactly what the generalize_eqs_vars variant does:

Coq < generalize_eqs_vars H.
1 subgoal

$m : \text{nat}$

$\text{gen}_x : \text{nat}$

$H : \text{Le gen}_x m$

----------------------------------

$\forall n : \text{nat}, \; \text{gen}_x = S \; n \rightarrow P \; n \; m$

Coq < induction H.
2 subgoals

$n : \text{nat}$

----------------------------------

$\forall n0 : \text{nat}, \; 0 = S \; n0 \rightarrow P \; n0 \; n$

subgoal 2 is:

$\forall n0 : \text{nat}, \; S \; n = S \; n0 \rightarrow P \; n0 \; (S \; m)$

As the hypothesis itself did not appear in the goal, we did not need to use an heterogeneous equality to relate the new hypothesis to the old one (which just disappeared here). However, the tactic works just a well in this case, e.g.:

Coq < Goal $\forall n \; m \; (p : \text{Le} \; (S \; n) \; m), \; Q \; (S \; n) \; m \; p$.
1 subgoal

----------------------------------

$\forall (n \; m : \text{nat}) \; (p : \text{Le} \; (S \; n) \; m), \; Q \; (S \; n) \; m \; p$

Coq < intros n m p ; generalize_eqs_vars p.
1 subgoal

$m : \text{nat}$

$\text{gen}_x : \text{nat}$

$p : \text{Le gen}_x m$

----------------------------------

$\forall (n : \text{nat}) \; (p0 : \text{Le} \; (S \; n) \; m),$

$\text{gen}_x = S \; n \rightarrow [p : (\text{Le gen}_x m)] = [p0 : (\text{Le} \; (S \; n) \; m)] \rightarrow Q \; (S \; n) \; m \; p0$

One drawback of this approach is that in the branches one will have to substitute the equalities back into the instance to get the right assumptions. Sometimes injection of constructors will also be needed to recover the needed equalities. Also, some subgoals should be directly solved because of inconsistent
Detailed examples of tactics

Three contexts arising from the constraints on indices. The nice thing is that we can make a tactic based on discriminate, injection and variants of substitution to automatically do such simplifications (which may involve the K axiom). This is what the `simplify_dep_elim` tactic from `Coq.Program.Equality` does. For example, we might simplify the previous goals considerably:

```
Coq < induction p ; simplify_dep_elim.
1 subgoal
  n0 : nat
  m : nat
  p : Le n0 m
  IHp : forall (n : nat) (p0 : Le (S n) m),
       n0 = S n -> [p : (Le n0 m)]=[p0 : (Le (S n) m)] -> Q (S n) m p0
  ============================
  Q (S n0) (S m) (LeS n0 m p)
  ``

The higher-order tactic `do_depind` defined in `Coq.Program.Equality` takes a tactic and combines the building blocks we’ve seen with it: generalizing by equalities calling the given tactic with the generalized induction hypothesis as argument and cleaning the subgoals with respect to equalities. Its most important instantiations are dependent induction and dependent destruction that do induction or simply case analysis on the generalized hypothesis. For example we can redo what we’ve done manually with dependent destruction:

```
Coq < Lemma ex : forall n m:nat, Le (S n) m -> P n m.
Coq < intros n m H.
Coq < dependent destruction H.
1 subgoal
  n : nat
  m : nat
  H : Le n m
  ============================
  P n (S m)
  ``

This gives essentially the same result as inversion. Now if the destructed hypothesis actually appeared in the goal, the tactic would still be able to invert it, contrary to dependent inversion. Consider the following example on vectors:

```
Coq < Set Implicit Arguments.
Coq < Variable A : Set.
Coq < Inductive vector : nat -> Type :=
  vnil : vector 0
  vcons : A -> forall n, vector n -> vector (S n).
Coq < Goal forall n, forall v : vector (S n),
  exists v' : vector n, exists a : A, v = vcons a v'.
Coq < intros n v.
```
Coq < dependent destruction v.
1 subgoal

\begin{align*}
n & : \text{nat} \\
a & : A \\
v & : \text{vector } n \\
\end{align*}

In this case, the \( v \) variable can be replaced in the goal by the generalized hypothesis only when it has a type of the form \( \text{vector } (S \ n) \), that is only in the second case of the \texttt{destruct}. The first one is dismissed because \( S \ n <> 0 \).

## 10.6.1 A larger example

Let's see how the technique works with \texttt{induction} on inductive predicates on a real example. We will develop an example application to the theory of simply-typed lambda-calculus formalized in a dependently-typed style:

Coq < Inductive type : Type :=
Coq < | base : type
Coq < | arrow : type -> type -> type.
Coq < Notation " t -> t' " := (arrow t t') (at level 20, t' at next level).
Coq < Inductive ctx : Type :=
Coq < | empty : ctx
Coq < | snoc : ctx -> type -> ctx.
Coq < Notation " G , tau " := (snoc G tau) (at level 20, t at next level).
Coq < Fixpoint conc (G D : ctx) : ctx :=
Coq < match D with
Coq < | empty => G
Coq < | snoc D' x => snoc (conc G D') x
Coq < end.
Coq < Notation " G ; D " := (conc G D) (at level 20).

We have defined types and contexts which are snoc-lists of types. We also have a \texttt{conc} operation that concatenates two contexts. The \texttt{term} datatype represents in fact the possible typing derivations of the calculus, which are isomorphic to the well-typed terms, hence the name. A term is either an application of:

- the axiom rule to type a reference to the first variable in a context,
- the weakening rule to type an object in a larger context
• the abstraction or lambda rule to type a function

• the application to type an application of a function to an argument

Once we have this datatype we want to do proofs on it, like weakening:

\[\text{Coq < Lemma weakening : } \forall G \ D \ \tau, \ \text{term (} G ; \ D \text{) } \tau \rightarrow \text{ Coq < } \forall \tau', \ \text{term (} G , \ \tau' ; \ D \text{) } \tau.\]

The problem here is that we can’t just use \textit{induction} on the typing derivation because it will forget about the \( G ; D \) constraint appearing in the instance. A solution would be to rewrite the goal as:

\[\text{Coq < Lemma weakening’ : } \forall G' \ \tau, \ \text{term } G' \ \tau \rightarrow \text{ Coq < } \forall G \ D, \ (G ; D) = G' \rightarrow \text{ Coq < } \forall \tau', \ \text{term (} G, \ \tau' ; \ D \text{) } \tau.\]

With this proper separation of the index from the instance and the right induction loading (putting \( G \) and \( D \) after the inducted-on hypothesis), the proof will go through, but it is a very tedious process. One is also forced to make a wrapper lemma to get back the more natural statement. The \textit{dependent induction} tactic alleviates this trouble by doing all of this plumbing of generalizing and substituting back automatically. Indeed we can simply write:

\[\text{Coq < Require Import Coq.Program.Tactics.} \]
\[\text{Coq < Lemma weakening : } \forall G \ D \ \tau, \ \text{term (} G ; \ D \text{) } \tau \rightarrow \text{ Coq < } \forall \tau', \ \text{term (} G , \ \tau' ; \ D \text{) } \tau.\]
\[\text{Coq < Proof with simpl in * ; simpl_depind ; auto.} \]
\[\text{Coq < intros G D tau H. dependent induction H generalizing G D ; intros.} \]

This call to \textit{dependent induction} has an additional argument which is a list of variables appearing in the instance that should be generalized in the goal, so that they can vary in the induction hypotheses. By default, all variables appearing inside constructors (except in a parameter position) of the instantiated hypothesis will be generalized automatically but one can always give the list explicitly.

\[\text{Coq < Show.} \]
\[\text{4 subgoals} \]
\[G : \text{ctx} \]
\[\tau : \text{type} \]
\[G0 : \text{ctx} \]
\[D : \text{ctx} \]
\[H : G, \ \tau = G0; D \]
\[\tau' : \text{type} \]
\[\text{term ((}G0, \ \tau'\text{);} D\text{) } \tau \]
\[\text{subgoal 2 is:} \]
\[\text{term ((}G0, \ \tau'0\text{);} D\text{) } \tau \]
\[\text{subgoal 3 is:} \]
\[\text{term ((}G0, \ \tau'0\text{);} D\text{) } (\tau \rightarrow \tau') \]
\[\text{subgoal 4 is:} \]
\[\text{term ((}G0, \ \tau'0\text{);} D\text{) } \tau' \]

Coq Reference Manual, V8.2pl2, June 29, 2010
The simpl_depind tactic includes an automatic tactic that tries to simplify equalities appearing at the beginning of induction hypotheses, generally using trivial applications of reflexivity. In cases where the equality is not between constructor forms though, one must help the automation by giving some arguments, using the specialize tactic.

Coq < destruct D... apply weak ; apply ax. apply ax.
Coq < destruct D...
Coq < Show.
4 subgoals

\[ G : ctx \]
\[ \tau : type \]
\[ H : \text{term} G \tau \]
\[ \tau' : type \]
\[ \text{IHterm} : \forall G0 D : ctx, \]
\[ \quad G = G0; D \to \forall \tau' : \text{type}, \text{term} ((G0, \tau'); D) \tau \]
\[ \tau'0 : type \]
\[ \text{============================} \]
\[ \text{term} ((G, \tau'), \tau'0) \tau \]
subgoal 2 is:
\[ \text{term} (((G0, \tau'0); D), t) \tau \]
subgoal 3 is:
\[ \text{term} ((G0, \tau'0); D) (\tau \to \tau') \]
subgoal 4 is:
\[ \text{term} ((G0, \tau'0); D) \tau' \]
Coq < specialize (IHterm G empty).
4 subgoals

\[ G : ctx \]
\[ \tau : type \]
\[ H : \text{term} G \tau \]
\[ \tau' : type \]
\[ \text{IHterm} : G = G; \text{empty} \to \forall \tau' : \text{type}, \text{term} ((G, \tau'); \text{empty}) \tau \]
\[ \tau'0 : type \]
\[ \text{============================} \]
\[ \text{term} ((G, \tau'), \tau'0) \tau \]
subgoal 2 is:
\[ \text{term} (((G0, \tau'0); D), t) \tau \]
subgoal 3 is:
\[ \text{term} ((G0, \tau'0); D) (\tau \to \tau') \]
subgoal 4 is:
\[ \text{term} ((G0, \tau'0); D) \tau' \]

Then the automation can find the needed equality \( G = G \) to narrow the induction hypothesis further. This concludes our example.

Coq < simpl_depind.
4 subgoals

\[ G : ctx \]
\[ \tau : type \]
See also: The induction 11, case 9 and inversion 8.10 tactics.

## 10.7 autorewrite

Here are two examples of autorewrite use. The first one (Ackermann function) shows actually a quite basic use where there is no conditional rewriting. The second one (Mac Carthy function) involves conditional rewritings and shows how to deal with them using the optional tactic of the \texttt{Hint Rewrite} command.

### Example 1: Ackermann function

Coq < Reset Initial.
Coq < Require Import Arith.
Coq < Axiom Ack0 : forall m:nat, Ack 0 m = S m.
Coq < Axiom Ack1 : forall n:nat, Ack (S n) 0 = Ack n 1.
Coq < Axiom Ack2 : forall n m:nat, Ack (S n) (S m) = Ack n (Ack (S n) m).
Coq < Hint Rewrite Ack0 Ack1 Ack2 : base0.
Coq < Lemma ResAck0 :
Coq < Ack 3 2 = 29.
1 subgoal

```
  Ack 3 2 = 29
```

Coq < autorewrite with base0 using try reflexivity.
Proof completed.

### Example 2: Mac Carthy function

Coq < Require Import Omega.
Coq < Variable g :
Coq < nat -> nat -> nat.
Coq < Axiom g0 :
Coq <   forall m:nat, g 0 m = m.
Coq < Axiom
Coq <   g1 :
Coq <   forall n m:nat,
Coq <       (n > 0) -> (m > 100) -> g n m = g (pred n) (m - 10).
Coq < Axiom
Coq <   g2 :
Coq <   forall n m:nat,
Coq <       (n > 0) -> (m <= 100) -> g n m = g (S n) (m + 11).

Coq < Hint Rewrite g0 g1 g2 using omega : base1.
Coq < Lemma Resg0 :
Coq <   g 1 110 = 100.
   1 subgoal

            110 = 100
Coq < autorewrite with base1 using reflexivity || simpl.
Proof completed.

Coq < Lemma Resg1 : g 1 95 = 91.
   1 subgoal

            91 = 91
Coq < autorewrite with base1 using reflexivity || simpl.
Proof completed.

10.8 quote

The tactic quote allows to use Barendregt’s so-called 2-level approach without writing any ML code. Suppose you have a language \( L \) of 'abstract terms' and a type \( A \) of 'concrete terms' and a function \( f : L \rightarrow A \). If \( L \) is a simple inductive datatype and \( f \) a simple fixpoint, quote \( f \) will replace the head of current goal by a convertible term of the form \((f \ t)\). \( L \) must have a constructor of type: \( A \rightarrow L \).

Here is an example:

Coq < Require Import Quote.
Coq < Parameters A B C : Prop.
   A is assumed
   B is assumed
   C is assumed

Coq < Inductive formula : Type :=
Coq <   | f_and : formula -> formula -> formula (* binary constructor *)
Coq <   | f_or : formula -> formula -> formula
Coq <   | f_not : formula -> formula (* unary constructor *)
Coq <   | f_true : formula (* 0-ary constructor *)
Coq <  | f_const : Prop -> formula (* constructor for constants *).
formulas is defined
formula_rect is defined
formula_ind is defined
formula_rec is defined

Coq < Fixpoint interp_f (f: formula) : Prop :=
Coq < FIXPOINT interp_f is recursively defined (decreasing on 1st argument)

Coq < quote interp_f.
quote interp_f is not a simple fixpoint

The algorithm to perform this inversion is: try to match the term with right-hand sides expression of 
f. If there is a match, apply the corresponding left-hand side and call yourself recursively on sub-terms. 
If there is no match, we are at a leaf: return the corresponding constructor (here f_const) applied to 
the term.

Error messages:

1. quote: not a simple fixpoint
   Happens when quote is not able to perform inversion properly.

10.8.1 Introducing variables map

The normal use of quote is to make proofs by reflection: one defines a function simplify : 
formula -> formula and proves a theorem simplify_ok: (f:formula)(interp_f 
(simplify f)) -> (interp_f f). Then, one can simplify formulas by doing:

quote interp_f.
apply simplify_ok.
compute.

But there is a problem with leaves: in the example above one cannot write a function that implements, 
for example, the logical simplifications $A \land A \rightarrow A$ or $A \land \neg A \rightarrow \text{False}$. This is because the Prop is 
impredicative.
It is better to use that type of formulas:

\[ \text{Coq} \langle \text{Inductive formula : Set :=} \]
\[ \text{Coq} \langle | f\_and : formula -> formula -> formula} \]
\[ \text{Coq} \langle | f\_or : formula -> formula -> formula} \]
\[ \text{Coq} \langle | f\_not : formula -> formula} \]
\[ \text{Coq} \langle | f\_true : formula} \]
\[ \text{Coq} \langle | f\_atom : index -> formula.} \]

\text{formula is defined}
\text{formula_rect is defined}
\text{formula_ind is defined}
\text{formula_rec is defined}

\text{index is defined in module quote. Equality on that type is decidable so we are able to simplify} \ A \land \ A \ into \ A \ at \ the \ abstract \ level.\\n
When there are variables, there are bindings, and \text{quote} provides also a type \{\text{varmap } A\} of bindings from \text{index} to any set \text{A}, and a function \text{varmap_find} to search in such maps. The interpretation function has now another argument, a variables map:

\[ \text{Coq} \langle \text{Fixpoint interp\_f (vm:} \]
\[ \text{Coq} \langle \text{varmap Prop)} (f:formula)} \ (\text{struct f} : \text{Prop :=} \]
\[ \text{Coq} \langle \text{match f with} \]
\[ \text{Coq} \langle | \ f\_and f1 f2 => interp\_f vm f1 \land interp\_f vm f2 \]
\[ \text{Coq} \langle | \ f\_or f1 f2 => interp\_f vm f1 \lor interp\_f vm f2 \]
\[ \text{Coq} \langle | \ f\_not f1 => \neg interp\_f vm f1 \]
\[ \text{Coq} \langle | \ f\_true => \text{True} \]
\[ \text{Coq} \langle | \ f\_atom i => \text{varmap\_find } \text{True } i \text{ vm} \]
\[ \text{Coq} \langle \text{end.} \]

\text{interp\_f is recursively defined (decreasing on 2nd argument)}\\n
\text{quote handles this second case properly:}\\n
\[ \text{Coq} \langle \text{Goal } A \land (B \lor A) \land (A \lor \neg B).} \]
\[ \text{1 subgoal} \]
\[ \text{================================================================================} \]
\[ A \land (B \lor A) \land (A \lor \neg B) \]
\[ \text{Coq} \langle \text{quote interp\_f.} \]
\[ \text{1 subgoal} \]
\[ \text{================================================================================} \]
\[ \text{interp\_f} \]
\[ (\text{Node}_vm B (\text{Node}_vm A (\text{Empty}_vm Prop) (\text{Empty}_vm Prop)) (\text{Empty}_vm Prop)) \]
\[ (f\_and (f\_atom (\text{Left\_idx } \text{End\_idx})) \]
\[ (f\_and (f\_or (f\_atom \text{End\_idx}) (f\_atom (\text{Left\_idx } \text{End\_idx})))) \]
\[ (f\_or (f\_atom (\text{Left\_idx } \text{End\_idx})) (f\_not (f\_atom \text{End\_idx})))) \]

\text{It builds } \text{vm} \text{ and } \text{t} \text{ such that } \text{(f} \text{ vm} \text{ t}) \text{ is convertible with the conclusion of current goal.}\\n
\textbf{10.8.2 Combining variables and constants}\\n
One can have both variables and constants in abstracts terms; that is the case, for example, for the \text{ring} tactic (chapter 23). Then one must provide to \text{quote} a list of \text{constructors of constants}. For example,
if the list is \([0 \ S]\) then closed natural numbers will be considered as constants and other terms as variables.

Example:

```coq
Coq < Inductive formula : Type :=
Coq <  | f_and : formula \rightarrow formula \rightarrow formula
Coq <  | f_or : formula \rightarrow formula \rightarrow formula
Coq <  | f_not : formula \rightarrow formula
Coq <  | f_true : formula
Coq <  | f_const : Prop \rightarrow formula (* constructor for constants *)
Coq <  | f_atom : index \rightarrow formula.
Coq < Fixpoint interp_f
Coq <  (vm: (* constructor for variables *)
Coq <  varmap Prop) (f:formula) {struct f} : Prop :=
Coq <  match f with
Coq <  | f_and f1 f2 => interp_f vm f1 \ /
Coq <  \ interp_f vm f2
Coq <  | f_or f1 f2 => interp_f vm f1 \ /
Coq <  \ interp_f vm f2
Coq <  | f_not f1 => \ interp_f vm f1
Coq <  | f_true => True
Coq <  | f_const c => c
Coq <  | f_atom i => varmap_find True i vm
Coq < end.
Coq < Goal
Coq < A \ /
Coq <  (A \ / True) \ /
Coq <  \ ~ B \ /
Coq <  \ (C \ <-> C).

Coq < quote interp_f [ A B ].
1 subgoal

```

```coq
interp_f (Node_vm (C <-> C) (Empty_vm Prop) (Empty_vm Prop))
(f_and (f_const A)
 (f_and (f_or (f_const A) f_true)
  (f_and (f_not (f_const B)) (f_atom End_id)))

Coq < Undo.
1 subgoal

```

```coq
A \ /
Coq <  (A \ / True) \ /
Coq <  \ ~ B \ /
Coq <  \ (C \ <-> C)

Coq < quote interp_f [ B C iff ].
1 subgoal

```

```coq
interp_f (Node_vm A (Empty_vm Prop) (Empty_vm Prop))
(f_and (f_atom End_id)
 (f_and (f_or (f_atom End_id) f_true)
  (f_and (f_not (f_const B)) (f_const (C <-> C)))

```

**Warning:** Since function inversion is undecidable in general case, don’t expect miracles from it!

**See also:** comments of source file tactics/contrib/polynom/quote.ml

**See also:** the ring tactic (Chapter 23)
10.9 Using the tactical language

10.9.1 About the cardinality of the set of natural numbers

A first example which shows how to use the pattern matching over the proof contexts is the proof that natural numbers have more than two elements. The proof of such a lemma can be done as follows:

Coq < Lemma card_nat :
Coq < ~ (exists x : nat, exists y : nat, forall z:nat, x = z \/ y = z).
Coq < Proof.
Coq < red; intros (x, (y, Hy)).
Coq < elim (Hy 0); elim (Hy 1); elim (Hy 2); intros;
Coq < match goal with
Coq < | [ _:(?a = ?b), _:(?a = ?c) ] =>
Coq < cut (b = c); [ discriminate | apply trans_equal with a; auto ]
Coq < end.
Coq < Qed.

We can notice that all the (very similar) cases coming from the three eliminations (with three distinct natural numbers) are successfully solved by a match goal structure and, in particular, with only one pattern (use of non-linear matching).

10.9.2 Permutation on closed lists

Another more complex example is the problem of permutation on closed lists. The aim is to show that a closed list is a permutation of another one.

First, we define the permutation predicate as shown in table 10.1.

A more complex example is the problem of permutation on closed lists. The aim is to show that a closed list is a permutation of another one. First, we define the permutation predicate as shown on Figure 10.1.

Next, we can write naturally the tactic and the result can be seen on Figure 10.2. We can notice that we use two toplevel definitions PermutProve and Permut. The function to be called is PermutProve which computes the lengths of the two lists and calls Permut with the length if the two lists have the same length. Permut works as expected. If the two lists are equal, it concludes. Otherwise, if the lists have identical first elements, it applies Permut on the tail of the lists. Finally, if the
Detailed examples of tactics

Figure 10.2: Permutation tactic

lists have different first elements, it puts the first element of one of the lists (here the second one which appears in the `permut` predicate) at the end if that is possible, i.e., if the new first element has been at this place previously. To verify that all rotations have been done for a list, we use the length of the list as an argument for `Permut` and this length is decremented for each rotation down to, but not including, 1 because for a list of length \( n \), we can make exactly \( n - 1 \) rotations to generate at most \( n \) distinct lists. Here, it must be noticed that we use the natural numbers of COQ for the rotation counter. On Figure 9.1, we can see that it is possible to use usual natural numbers but they are only used as arguments for primitive tactics and they cannot be handled, in particular, we cannot make computations with them. So, a natural choice is to use COQ data structures so that COQ makes the computations (reductions) by `eval compute in` and we can get the terms back by `match`.

With `PermutProve`, we can now prove lemmas as follows:

```coq
Ltac Permut n :=
  match goal with
  | |- (permut _ ?l ?l) ⇒ apply permut_refl
  | |- (permut _ (?a :: ?l1) (?a :: ?l2)) ⇒
    let newn := eval compute in (length l1) in
    (apply permut_cons; Permut newn)
  | |- (permut ?A (?a :: ?l1) ?l2) ⇒
    match eval compute in n with
    | 1 ⇒ fail
    | _ ⇒
      let l1′ := constr:(l1 ++ a :: nil) in
      (apply (permut_trans A (a :: l1) l1′ l2);
       [ apply permut_append | compute; Permut (pred n) ])
  end.
Permut is defined

Ltac PermutProve :=
  match goal with
  | |- (permut _ ?l1 ?l2) ⇒
    match eval compute in (length l1 = length l2) with
    | (?n = ?n) ⇒ Permut n
    | _ ⇒
      end.
PermutProve is defined
```

```coq
Lemma permut_ex1 :
  permut nat (1 :: 2 :: 3 :: nil) (3 :: 2 :: 1 :: nil).

Lemma permut_ex2 :
  permut nat
  (0 :: 1 :: 2 :: 3 :: 4 :: 5 :: 6 :: 7 :: 8 :: 9 :: nil)
  (0 :: 2 :: 4 :: 6 :: 8 :: 9 :: 7 :: 5 :: 3 :: 1 :: nil).
```
10.9 Using the tactical language

10.9.3 Deciding intuitionistic propositional logic

The pattern matching on goals allows a complete and so a powerful backtracking when returning tactic values. An interesting application is the problem of deciding intuitionistic propositional logic. Considering the contraction-free sequent calculi $LJT^*$ of Roy Dyckhoff ([54]), it is quite natural to code such a tactic using the tactic language as shown on Figures 10.3 and 10.4. The tactic $Axioms$ tries to conclude using usual axioms. The tactic $DSimplif$ applies all the reversible rules of Dyckhoff’s system. Finally, the tactic $TautoProp$ (the main tactic to be called) simplifies with $DSimplif$, tries to conclude with $Axioms$ and tries several paths using the backtracking rules (one of the four Dyckhoff’s rules for the left implication to get rid of the contraction and the right or).

For example, with $TautoProp$, we can prove tautologies like those:

\[
\text{Coq} \quad \text{Lemma tauto_ex1} : \quad \forall A, B : \text{Prop}, \quad A \land B \rightarrow A \lor B.
\]

\[
\text{Coq} \quad \text{Proof.} \quad \text{TautoProp. Qed.}
\]

\[
\text{Coq} \quad \text{Lemma tauto_ex2} : \quad \\
\text{Coq} \quad \forall A, B : \text{Prop}, \quad \neg \neg B \rightarrow B \rightarrow (A \rightarrow B) \rightarrow \neg \neg A \rightarrow B.
\]

\[
\text{Coq} \quad \text{Proof.} \quad \text{TautoProp. Qed.}
\]

10.9.4 Deciding type isomorphisms

A more tricky problem is to decide equalities between types and modulo isomorphisms. Here, we choose to use the isomorphisms of the simply typed $\lambda$-calculus with Cartesian product and unit type (see, for example, [43]). The axioms of this $\lambda$-calculus are given by table 10.5.

A more tricky problem is to decide equalities between types and modulo isomorphisms. Here, we choose to use the isomorphisms of the simply typed $\lambda$-calculus with Cartesian product and unit type (see, for example, [43]). The axioms of this $\lambda$-calculus are given on Figure 10.5.

The tactic to judge equalities modulo this axiomatization can be written as shown on Figures 10.6 and 10.7. The algorithm is quite simple. Types are reduced using axioms that can be oriented (this done by $MainSimplif$). The normal forms are sequences of Cartesian products without Cartesian product in the left component. These normal forms are then compared modulo permutation of the components (this is done by $CompareStruct$). The main tactic to be called and realizing this algorithm is $IsoProve$.

Here are examples of what can be solved by $IsoProve$.

\[
\text{Coq} \quad \text{Lemma isos_ex1} : \quad \forall A, B : \text{Set}, \quad A \times \text{unit} \times B = B \times (\text{unit} \times A).
\]

\[
\text{Coq} \quad \text{Proof.} \quad \text{intros; IsoProve.}
\]

\[
\text{Coq} \quad \text{Ltac Axioms :=} \\
\text{Coq} \quad \text{match goal with} \\
\text{Coq} \quad | \_ |- \text{True} => \text{trivial} \\
\text{Coq} \quad | \_ : \text{False} |- \_ => \text{elimtype False; assumption} \\
\text{Coq} \quad | \_ : ?A |- \_ => \text{auto} \\
\text{Coq} \quad \text{end.} \\
\text{Axioms is defined}
\]

Figure 10.3: Deciding intuitionistic propositions (1)
Coq < Ltac DSimplif :=
Coq <  repeat
Coq <  (intros;
Coq <  match goal with
Coq <    | id:(~ _) |- _ => red in id
Coq <    | id:(_ \ \ _) |- _ =>
Coq <      elim id; do 2 intro; clear id
Coq <    | id:(_ \ _ ) |- _ =>
Coq <      elim id; intro; clear id
Coq <    | id:(_ / _ ) |- _ =>
Coq <      elim id; intro; clear id
Coq <    | id:(_ / _ ) |- _ =>
Coq <      elim id; intro; clear id
Coq <    | id:(_ / _ ) |- _ =>
Coq <      elim id; intro; clear id
Coq <    | id:(_ / _ ) |- _ =>
Coq <      elim id; intro; clear id
Coq <    | id:(_ / _ ) |- _ =>
Coq <      elim id; intro; clear id
Coq <    | id0:(?A -> ?B),id1:?A |- _ =>
Coq <      cut B; [ intro; clear id0 | apply id0; assumption ]
Coq <    | |- (_ / _) => split
Coq <    | |- (~ _) => split
Coq <    | |- (~ _) => red
Coq <  end).
DSimplif is defined

Coq < Ltac TautoProp :=
Coq < DSimplif;
Coq < Axioms ||
Coq < match goal with
Coq <    | id:((?A -> ?B) -> ?C) |- _ =>
Coq <      cut (B -> C);
Coq <    | intro; cut (A -> B);
Coq <      [ intro; cut C;
Coq <      | intro; clear id | apply id; assumption ]
Coq <    | intro; apply id; intro; assumption ]; TautoProp
Coq <    | id:(~ ?A -> ?B) |- _ =>
Coq <      cut (False -> B);
Coq <    | intro; cut (A -> False);
Coq <      [ intro; cut B;
Coq <      | intro; clear id | apply id; assumption ]
Coq <    | intro; apply id; red; intro; assumption ]; TautoProp
Coq <    | |- (_ \ _ ) => (left; TautoProp) || (right; TautoProp)
Coq <  end.
TautoProp is defined

Figure 10.4: Deciding intuitionistic propositions (2)

Coq < Qed.

Coq <
Coq < Lemma isos_ex2 :
Coq <  forall A B C:Set,
10.9 Using the tactical language

Coq < Open Scope type_scope.
Coq < Section Iso_axioms.
Coq < Variables A B C : Set.
Coq < Axiom Com : A * B = B * A.
Coq < Axiom Ass : A * (B * C) = A * B * C.
Coq < Axiom P_unit : A * unit = A.
Coq < Axiom AR_unit : (A -> unit) = unit.
Coq < Axiom AL_unit : (unit -> A) = A.
Coq < Lemma Cons : B = C -> A * B = A * C.
Coq < Proof.
Coq < intro Heq; rewrite Heq; apply refl_equal.
Coq < Qed.
Coq < End Iso_axioms.

Figure 10.5: Type isomorphism axioms

Coq < (A * unit -> B * (C * unit)) =
Coq < Proof.
Coq < intros; IsoProve.
Coq < Qed.
Coq < Ltac DSimplif trm :=
Coq < match trm with
Coq < | (?A * ?B * ?C) =>
Coq < rewrite <- (Ass A B C); try MainSimplif
Coq < | (?A * ?B -> ?C) =>
Coq < rewrite (Cur A B C); try MainSimplif
Coq < | (?A -> ?B * ?C) =>
Coq < rewrite (Dis A B C); try MainSimplif
Coq < | (?A * unit) =>
Coq < rewrite (P_unit A); try MainSimplif
Coq < | (unit * ?B) =>
Coq < rewrite (Com unit B); try MainSimplif
Coq < | (?A -> unit) =>
Coq < rewrite (AR_unit A); try MainSimplif
Coq < | (unit -> ?B) =>
Coq < rewrite (AL_unit B); try MainSimplif
Coq < | (?A * ?B) =>
Coq < (DSimplif A; try MainSimplif) || (DSimplif B; try MainSimplif)
Coq < | (?A -> ?B) =>
Coq < (DSimplif A; try MainSimplif) || (DSimplif B; try MainSimplif)
Coq < end
Coq < with MainSimplif :=
Coq < match goal with
Coq < | |- (?A = ?B) => try DSimplif A; try DSimplif B
Coq < end.
DSimplif is defined
MainSimplif is defined

Coq < Ltac Length trm :=
Coq < match trm with
Coq < | (_ * ?B) => let succ := Length B in constr:(S succ)
Coq < | _ => constr:1
Coq < end.
Length is defined

Coq < Ltac assoc := repeat rewrite <- Ass.
assoc is defined

Figure 10.6: Type isomorphism tactic (1)
Ltac DoCompare n :=
match goal with
  | [ |- (?A = ?A) ] => apply refl_equal
    apply Cons; let newn := Length B in
    DoCompare newn
  | [ |- (?A * ?B = ?C) ] =>
    match eval compute in n with
    | 1 => fail
    | _ =>
      pattern (A * B) at 1; rewrite Com; assoc; DoCompare (pred n)
    end
end.

DoCompare is defined

Ltac CompareStruct :=
match goal with
  | [ |- (?A = ?B) ] =>
    let l1 := Length A
    with l2 := Length B in
    match eval compute in (l1 = l2) with
    | (?n = ?n) => DoCompare n
    end
end.

CompareStruct is defined

Ltac IsoProve := MainSimplif; CompareStruct.

IsoProve is defined

Figure 10.7: Type isomorphism tactic (2)
Chapter 11

The C-zar mathematical proof language

11.1 Introduction

11.1.1 Foreword

In this chapter, we describe an alternative language that may be used to do proofs using the Coq proof assistant. The language described here uses the same objects (proof-terms) as Coq, but it differs in the way proofs are described. This language was created by Pierre Corbineau at the Radboud University of Nijmegen, The Netherlands.

The intent is to provide a language where proofs are less formalism- and implementation-sensitive, and in the process to ease a bit the learning of computer-aided proof verification.

11.1.2 What is a declarative proof?

In vanilla Coq, proofs are written in the imperative style: the user issues commands that transform a so called proof state until it reaches a state where the proof is completed. In the process, the user mostly described the transitions of this system rather than the intermediate states it goes through.

The purpose of a declarative proof language is to take the opposite approach where intermediate states are always given by the user, but the transitions of the system are automated as much as possible.

While not being a purely declarative language, the C-zar mathematical proof language aims at providing a solution for users who wish to edit Coq proofs following the declarative philosophy.

11.1.3 Well-formedness and Completeness

The C-zar mathematical proof language introduces a notion of well-formed proofs which are weaker than correct (and complete) proofs. Well-formed proofs are actually proof script where only the reasoning is incomplete. All the other aspects of the proof are correct:

- All objects referred to exist where they are used
- Conclusion steps actually prove something related to the conclusion of the theorem (the thesis).
- Hypothesis introduction steps are done when the goal is an implication with a corresponding assumption.
- Sub-objects in the elimination steps for tuples are correct sub-objects of the tuple being decomposed.
11 The C-zar mathematical proof language

- Patterns in case analysis are type-correct, and induction is well guarded.

11.1.4 Note for tactics users

This section explain what differences the casual Coq user will experience using the C-zar mathematical proof language.

1. The focusing mechanism is constrained so that only one goal at a time is visible.

2. Giving a statement that Coq cannot prove does not produce an error, only a warning: this allows to go on with the proof and fill the gap later.

3. Tactics can still be used for justifications and after escape.

11.1.5 Compatibility

The C-zar mathematical proof language is available for all Coq interfaces that use text-based interaction, including:

- the command-line toplevel coqtop
- the native GUI coqide
- the Proof-General emacs mode
- Cezary Kaliszyk’s Web interface
- L.E. Mamane’s tmEgg TeXmacs plugin

However it is not supported by structured editors such as PCoq.

11.2 Syntax

Here is a complete formal description of the syntax for C-zar commands.

The lexical conventions used here follows those of section 1.1.

Conventions:

- `<tactic>` stands for an Coq tactic.

11.2.1 Temporary names

In proof commands where an optional name is asked for, omitting the name will trigger the creation of a fresh temporary name (e.g. for a hypothesis). Temporary names always start with an underscore ‘_’ character (e.g. `_hyp0`). Temporary names have a lifespan of one command: they get erased after the next command. They can however be used safely in the step after their creation.
11.3 Language description

11.3.1 Starting and Ending a mathematical proof

The standard way to use the C-zar mathematical proof language is to first state a Lemma/Theorem/Definition and then use the proof command to switch the current subgoal to mathematical mode. After the proof is completed, the end proof command will close the mathematical proof. If any subgoal remains to be proved, they will be displayed using the usual Coq display.

Coq < Theorem this_is_trivial: True.
1 subgoal

  ============================
  True

Coq < proof.
The proof command only applies to one subgoal, thus if several sub-goals are already present, the `proof .. end proof` sequence has to be used several times.
1 subgoal

*** Declarative Mode ***

=--------------------
thesis := True
Coq < thus thesis.
Subproof completed, now type "end proof".
Coq < end proof.
Proof completed.

As with all other block structures, the end proof command assumes that your proof is complete. If not, executing it will be equivalent to admitting that the statement is proved: A warning will be issued and you will not be able to run the Qed command. Instead, you can run Admitted if you wish to start another theorem and come back later.

Coq < Theorem this_is_not_so_trivial: False.
1 subgoal

--------------------
False
Coq < proof.
1 subgoal

*** Declarative Mode ***

--------------------
thesis := False
Coq < end proof. (* here a warning is issued *)
Proof completed.
Coq < Qed. (* fails : the proof in incomplete *)
proof.
(* Some proof has been skipped here *)
end proof.
Error: Attempt to save an incomplete proof
Coq < Admitted. (* Oops! *)
this_is_not_so_trivial is assumed

11.3.2 Switching modes

When writing a mathematical proof, you may wish to use procedural tactics at some point. One way to do so is to write a using-phrase in a deduction step (see section 11.3.14). The other way is to use an escape...return block.

Coq < Show.
1 subgoal

*** Declarative Mode ***

--------------------

Coq Reference Manual, V8.2p2, June 29, 2010
11 The C-zar mathematical proof language

\[
\text{thesis} := \\
\text{True}
\]

Coq < escape.
1 subgoal

============================

\text{True}

Coq < auto.
Subgoal proved
Subproof completed, now type "return".

Coq < return.
Subproof completed, now type "end proof".

The return statement expects all subgoals to be closed, otherwise a warning is issued and the proof cannot be saved anymore.

It is possible to use the \text{proof} command inside an \text{escape...return} block, thus nesting a mathematical proof inside a procedural proof inside a mathematical proof ...

11.3.3 Computation steps

The \text{reconsider ... as} command allows to change the type of a hypothesis or of \text{thesis} to a convertible one.

Coq < Show.
1 subgoal

*** Declarative Mode ***

\[a := \text{false} : \text{bool}\]
\[b := \text{true} : \text{bool}\]
\[H : \text{if a then True else False}\]

\[
\text{thesis} := \\
\text{if b then True else False}
\]

Coq < reconsider \(H\) as \text{False}.
1 subgoal

*** Declarative Mode ***

\[a := \text{false} : \text{bool}\]
\[b := \text{true} : \text{bool}\]
\[H : \text{False}\]

\[
\text{thesis} := \\
\text{if b then True else False}
\]

Coq < reconsider \text{thesis} as \text{True}.
1 subgoal

*** Declarative Mode ***

\[a := \text{false} : \text{bool}\]
\[b := \text{true} : \text{bool}\]
\[H : \text{False}\]
11.3 Language description

11.3.4 Deduction steps

The most common instruction in a mathematical proof is the deduction step: it asserts a new statement (a formula/type of the pC\text{IC}) and tries to prove it using a user-provided indication: the justification. The asserted statement is then added as a hypothesis to the proof context.

Coq < Show.
1 subgoal
  *** Declarative Mode ***
  x : nat
  H : x = 2
  thesis :=
  2 + x = 4

Coq < have H’:(2+x=2+2) by H.
1 subgoal
  *** Declarative Mode ***
  x : nat
  H : x = 2
  H’ : 2 + x = 2 + 2
  thesis :=
  2 + x = 4

It is very often the case that the justifications uses the last hypothesis introduced in the context, so the then keyword can be used as a shortcut, e.g. if we want to do the same as the last example:

Coq < Show.
1 subgoal
  *** Declarative Mode ***
  x : nat
  H : x = 2
  thesis :=
  2 + x = 4

Coq < then (2+x=2+2).
1 subgoal
  *** Declarative Mode ***
  x : nat
  H : x = 2
  _fact : 2 + x = 2 + 2
  thesis :=
  2 + x = 4

In this example, you can also see the creation of a temporary name _fact.
11.3.5 Iterated equalities

A common proof pattern when doing a chain of deductions, is to do multiple rewriting steps over the same term, thus proving the corresponding equalities. The iterated equalities are a syntactic support for this kind of reasoning:

Coq < Show.
1 subgoal
  *** Declarative Mode ***

  x : nat
  H : x = 2
  thesis :=
  x + x = x * x

Coq < have (4 = 4).
1 subgoal
  *** Declarative Mode ***

  x : nat
  H : x = 2
  _fact : 4 = 4
  thesis :=
  x + x = x * x

Coq < =~ (2 + 2).
1 subgoal
  *** Declarative Mode ***

  x : nat
  H : x = 2
  _eq : 4 = 2 * 2
  thesis :=
  x + x = x * x

Coq < =~ (x * x) by H.
1 subgoal
  *** Declarative Mode ***

  x : nat
  H : x = 2
  _eq0 : 4 = x * x
  thesis :=
  x + x = x * x

Coq < =~ (2 + 2).
1 subgoal
  *** Declarative Mode ***
11.3 Language description

\_eq : 2 + 2 = x * x

\begin{verbatim}
thesis :=
  x + x = x * x
\end{verbatim}

Coq < =~ H':(x + x) by H.
1 subgoal

*** Declarative Mode ***

x : nat
H : x = 2
H' : x + x = x * x

\begin{verbatim}
thesis :=
  x + x = x * x
\end{verbatim}

Notice that here we use temporary names heavily.

11.3.6 Subproofs

When an intermediate step in a proof gets too complicated or involves a well contained set of interme-
diate deductions, it can be useful to insert its proof as a subproof of the current proof. this is done by
using the \texttt{claim ... end claim} pair of commands.

Coq < Show.
1 subgoal

*** Declarative Mode ***

x : nat
H : x + x = x * x

\begin{verbatim}
thesis :=
  x = 0 \or/ x = 2
\end{verbatim}

Coq < claim H':((x - 2) * x = 0).
1 subgoal

*** Declarative Mode ***

x : nat
H : x + x = x * x

\begin{verbatim}
thesis :=
  (x - 2) * x = 0
\end{verbatim}

A few steps later ...

Coq < thus thesis.
Warning: Insufficient justification.
Subproof completed, now type "end claim".

Coq < end claim.
1 subgoal

*** Declarative Mode ***
\( x : \text{nat} \)
\( H : x + x = x \times x \)
\( H' : (x - 2) \times x = 0 \)

\[ \text{thesis} := \]
\[ x = 0 \lor x = 2 \]

Now the rest of the proof can happen.

### 11.3.7 Conclusion steps

The commands described above have a conclusion counterpart, where the new hypothesis is used to refine the conclusion.

Let us begin with simple examples:

Coq < Show.

1 subgoal

### Declarative Mode ###

\( A : \text{Prop} \)
\( B : \text{Prop} \)
\( HA : A \)
\( HB : B \)

\[ \text{thesis} := \]
\[ A \lor B \]

Coq < hence B.

1 subgoal

### Declarative Mode ###

\( A : \text{Prop} \)
\( B : \text{Prop} \)
\( HA : A \)
\( HB : B \)
\( \_\text{fact} : B \)

\[ \text{thesis} := \]
\[ A \]

In the next example, we have to use thus because HB is no longer the last hypothesis.

Coq < Show.

1 subgoal

### Declarative Mode ###

<table>
<thead>
<tr>
<th>X</th>
<th>simple</th>
<th>with previous step</th>
<th>opens sub-proof</th>
<th>iterated equality</th>
</tr>
</thead>
<tbody>
<tr>
<td>intermediate step</td>
<td>have</td>
<td>then</td>
<td>claim</td>
<td>( \sim /\sim )</td>
</tr>
<tr>
<td>conclusion step</td>
<td>thus</td>
<td>hence</td>
<td>focus on</td>
<td>thus ( \sim /\sim )</td>
</tr>
</tbody>
</table>

Figure 11.2: Correspondence between basic forward steps and conclusion steps
The command fails the refinement process cannot find a place to fit the object in a proof of the conclusion.

The refinement process may induce non reversible choices, e.g. when proving a disjunction it may *choose* one side of the disjunction.

Coq < hence C. (* fails *)
Error: I could not relate this statement to the thesis.
In this example you can see that the right branch was chosen since $D$ remains to be proved.

Now for existential statements, we can use the `take` command to choose 2 as an explicit witness of existence.
It is also possible to prove the existence directly.

Coq < Show.
1 subgoal

*** Declarative Mode ***

P : nat -> Prop
HP : P 2
thesis :=
exists x : nat, P x

Coq < hence (P 2).
Subproof completed, now type "end proof".

Here a more involved example where the choice of \( P \) 2 propagates the choice of 2 to another part of the formula.

Coq < Show.
1 subgoal

*** Declarative Mode ***

P : nat -> Prop
R : nat -> nat -> Prop
HP : P 2
HR : R 0 2
thesis :=
exists x : nat, exists y : nat, P y \&\& R x y

Coq < thus (P 2) by HP.
1 subgoal

*** Declarative Mode ***

P : nat -> Prop
R : nat -> nat -> Prop
HP : P 2
HR : R 0 2
_fact : P 2
thesis :=
exists n : nat, R n 2

Now, an example with the **suffices** command. **suffices** is a sort of dual for **have**: it allows to replace the conclusion (or part of it) by a sufficient condition.
Coq < Show.
1 subgoal
  *** Declarative Mode ***
  A : Prop
  B : Prop
  P : nat -> Prop
  HP : forall x : nat, P x -> B
  HA : A

thesis :=
A, B

Coq < suffices to have x such that HP':(P x) to show B by HP,HP'.
1 subgoal
  *** Declarative Mode ***
  A : Prop
  B : Prop
  P : nat -> Prop
  HP : forall x : nat, P x -> B
  HA : A
  _cofact : forall x : nat, P x -> B

thesis :=
A, exists n : nat, P n

Finally, an example where focus is handy: local assumptions.

Coq < Show.
1 subgoal
  *** Declarative Mode ***
  A : Prop
  P : nat -> Prop
  HP : P 2
  HA : A

thesis :=
A, (forall x : nat, x = 2 -> P x)

Coq < focus on (forall x, x = 2 -> P x).
1 subgoal
  *** Declarative Mode ***
  A : Prop
  P : nat -> Prop
  HP : P 2
  HA : A

thesis :=
forall x : nat, x = 2 -> P x

Coq < let x be such that (x = 2).
1 subgoal
11.3 Language description

*** Declarative Mode ***

A : Prop
P : nat -> Prop
HP : P 2
HA : A
x : nat
_hyp : x = 2

thesis :=
P x

Coq < hence thesis by HP.
Subproof completed, now type "end focus".

Coq < end focus.
1 subgoal
*** Declarative Mode ***

A : Prop
P : nat -> Prop
HP : P 2
HA : A
_claim : forall x : nat, x = 2 -> P x

thesis :=
A

11.3.8 Declaring an Abbreviation

In order to shorten long expressions, it is possible to use the define ... as ... command to give a name to recurring expressions.

Coq < Show.
1 subgoal
*** Declarative Mode ***

x : nat
H : x = 0

thesis :=
x + x = x * x

Coq < define sqr x as (x * x).
1 subgoal
*** Declarative Mode ***

x : nat
H : x = 0
sqr := fun x : nat => x * x : nat -> nat

thesis :=
x + x = x * x

Coq < reconsider thesis as (x + x = sqr x).
1 1 subgoal
   *** Declarative Mode ***
   x : nat
H : x = 0
sqr := fun x : nat => x * x : nat -> nat
   =============================
thesis :=
x + x = sqr x

11.3.9 Introduction steps

When the thesis consists of a hypothetical formula (implication or universal quantification (e.g. A \rightarrow B)), it is possible to assume the hypothetical part A and then prove B. In the C-zar mathematical proof language, this comes in two syntactic flavors that are semantically equivalent: let and assume. Their syntax is designed so that let works better for universal quantifiers and assume for implications.

Coq < Show.
1 1 subgoal
   *** Declarative Mode ***
P : nat -> Prop
   =============================
thesis :=
   forall x : nat, P x -> P x
Coq < let x:nat.
1 1 subgoal
   *** Declarative Mode ***
P : nat -> Prop
x : nat
   =============================
thesis :=
P x -> P x
Coq < assume HP:(P x).
1 1 subgoal
   *** Declarative Mode ***
P : nat -> Prop
x : nat
HP : P x
   =============================
thesis :=
P x

In the let variant, the type of the assumed object is optional provided it can be deduced from the command. The objects introduced by let can be followed by assumptions using such that.

Coq < Show.
1 1 subgoal
   *** Declarative Mode ***
11.3 Language description

\[ P : \text{nat} \rightarrow \text{Prop} \]

\[ \text{thesis} := \]

\[ \forall x : \text{nat}, P \ x \rightarrow P \ x \]

Coq < let x. (* fails because x’s type is not clear *)

toplevel input, characters 4-5:
> let x.
> ^
ERROR: Cannot infer the type of x.

Coq < let x be such that HP:(P x). (* here x’s type is inferred from (P x) *)

1 subgoal

*** Declarative Mode ***

\[ P : \text{nat} \rightarrow \text{Prop} \]
\[ x : \text{nat} \]
\[ \text{HP} : P \ x \]

\[ \text{thesis} := \]

\[ P \ x \]

In the assume variant, the type of the assumed object is mandatory but the name is optional:

Coq < Show.

1 subgoal

*** Declarative Mode ***

\[ P : \text{nat} \rightarrow \text{Prop} \]
\[ x : \text{nat} \]

\[ \text{thesis} := \]

\[ P \ x \rightarrow P \ x \rightarrow P \ x \]

Coq < assume (P x). (* temporary name created *)

1 subgoal

*** Declarative Mode ***

\[ P : \text{nat} \rightarrow \text{Prop} \]
\[ x : \text{nat} \]
\[ \_hyp : P \ x \]

\[ \text{thesis} := \]

\[ P \ x \rightarrow P \ x \]

After such that, it is also the case:

Coq < Show.

1 subgoal

*** Declarative Mode ***

\[ P : \text{nat} \rightarrow \text{Prop} \]

\[ \text{thesis} := \]

\[ \forall x : \text{nat}, P \ x \rightarrow P \ x \]
The C-zar mathematical proof language

11.3.10 Tuple elimination steps

In the pC1c, many objects dealt with in simple proofs are tuples: pairs, records, existentially quantified formulas. These are so common that the C-zar mathematical proof language provides a mechanism to extract members of those tuples, and also objects in tuples within tuples within tuples...

Coq < let x be such that (P x). (* temporary name created *)
1 subgoal
  *** Declarative Mode ***

  P : nat -> Prop
  x : nat
  _hyp : P x
  thesis :=
  P x

Here is an example with pairs:

Coq < Show.
1 subgoal
  *** Declarative Mode ***

  p : nat * nat
  thesis :=
  fst p >= snd p \/ fst p <= snd p

Coq Reference Manual, V8.2pl2, June 29, 2010
1 subgoal
  *** Declarative Mode ***

\[ p : \text{nat} \times \text{nat} \]
\[ x : \text{nat} \]
\[ y : \text{nat} \]

\[ \text{thesis} := \]
\[ \text{fst}(x, y) \geq \text{snd}(x, y) \lor \text{fst}(x, y) < \text{snd}(x, y) \]

Coq < reconsider thesis as \( x \geq y \lor x < y \).
1 subgoal
  *** Declarative Mode ***

\[ p : \text{nat} \times \text{nat} \]
\[ x : \text{nat} \]
\[ y : \text{nat} \]

\[ \text{thesis} := \]
\[ x \geq y \lor x < y \]

It is sometimes desirable to combine assumption and tuple decomposition. This can be done using the \texttt{given} command.

Coq < Show.
1 subgoal
  *** Declarative Mode ***

\[ P : \text{nat} \rightarrow \text{Prop} \]
\[ \text{HP} : \forall n : \text{nat}, P n \rightarrow P (n - 1) \]

\[ \text{thesis} := \]
\[ (\exists m : \text{nat}, P m) \rightarrow P 0 \]

Coq < given \( m \) such that \( Hm \)(\( P m \)).
1 subgoal
  *** Declarative Mode ***

\[ P : \text{nat} \rightarrow \text{Prop} \]
\[ \text{HP} : \forall n : \text{nat}, P n \rightarrow P (n - 1) \]
\[ m : \text{nat} \]
\[ Hm : P m \]

\[ \text{thesis} := \]
\[ P 0 \]

11.3.11 Disjunctive reasoning

In some proofs (most of them usually) one has to consider several cases and prove that the \texttt{thesis} holds in all the cases. This is done by first specifying which object will be subject to case distinction (usually a disjunction) using \texttt{per cases}, and then specifying which case is being proved by using \texttt{suppose}.

Coq < \texttt{per cases on HAB}.
1 subgoal
*** Declarative Mode ***

A : Prop
B : Prop
C : Prop
HAC : A -> C
HBC : B -> C
HAB : A \lor B

thesis :=
C

Coq < suppose A.

1 subgoal
*** Declarative Mode ***

A : Prop
B : Prop
C : Prop
HAC : A -> C
HBC : B -> C
HAB : A \lor B
_hyp : A

thesis :=
C

Coq < hence thesis by HAC.

Subproof completed, now type "end cases" or start a new case.

Coq < suppose HB: B.

1 subgoal
*** Declarative Mode ***

A : Prop
B : Prop
C : Prop
HAC : A -> C
HBC : B -> C
HAB : A \lor B
HB : B

thesis :=
C

Coq < thus thesis by HB, HBC.

Subproof completed, now type "end cases" or start a new case.

Coq < end cases.

Subproof completed, now type "end proof".

The proof is well formed (but incomplete) even if you type end cases or the next suppose before the previous case is proved.

If the disjunction is derived from a more general principle, e.g. the excluded middle axiom), it is desirable to just specify which instance of it is being used:
Coq < Hypothesis EM : forall P:Prop, P \ / ~ P.
EM is assumed

Coq < per cases of (A \ / ~A) by EM.
1 subgoal
  *** Declarative Mode ***

EM : forall P : Prop, P \ / ~ P
A : Prop
C : Prop
HAC : A -> C
HNAC : ~ A -> C
anonymous_matched : A \ / ~ A
============================
thesis :=
C

Coq < suppose (~A).
1 subgoal
  *** Declarative Mode ***

EM : forall P : Prop, P \ / ~ P
A : Prop
C : Prop
HAC : A -> C
HNAC : ~ A -> C
anonymous_matched : A \ / ~ A
_hyp : ~ A
============================
thesis :=
C

Coq < hence thesis by HNAC.
Subproof completed, now type "end cases" or start a new case.

Coq < suppose A.
1 subgoal
  *** Declarative Mode ***

EM : forall P : Prop, P \ / ~ P
A : Prop
C : Prop
HAC : A -> C
HNAC : ~ A -> C
anonymous_matched : A \ / ~ A
_hyp : A
============================
thesis :=
C

Coq < hence thesis by HAC.
Subproof completed, now type "end cases" or start a new case.

Coq < end cases.
Subproof completed, now type "end proof".
11.3.12 Proofs per cases

If the case analysis is to be made on a particular object, the script is very similar: it starts with \texttt{per cases on object} instead.

\begin{verbatim}
Coq < per cases on (EM A).
1 subgoal
*** Declarative Mode ***

EM : forall P : Prop, P \lor \neg P
A : Prop
C : Prop
HAC : A \rightarrow C
HNAC : \neg A \rightarrow C

thesis :=
C

Coq < suppose (~A).
1 subgoal
*** Declarative Mode ***

EM : forall P : Prop, P \lor \neg P
A : Prop
C : Prop
HAC : A \rightarrow C
HNAC : \neg A \rightarrow C
_hyp : \neg A

thesis :=
C

If the object on which a case analysis occurs in the statement to be proved, the command \texttt{suppose it is pattern} is better suited than \texttt{suppose}. \texttt{pattern} may contain nested patterns with \texttt{as} clauses. A detailed description of patterns is to be found in figure 1.2. here is an example.

Coq < per cases on x.
1 subgoal
*** Declarative Mode ***

A : Prop
B : Prop
x : bool

thesis :=
(if x then A else B) \rightarrow A \lor B

Coq < suppose it is true.
1 subgoal
*** Declarative Mode ***

A : Prop
B : Prop
x : bool
\end{verbatim}
\[
\text{thesis} := \ A \rightarrow A \lor B
\]

Coq < assume A.
1 subgoal

*** Declarative Mode ***

A : Prop
B : Prop
x : bool
_hyp : A

\[
\text{thesis} := \ A \lor B
\]

Coq < hence A.
Subproof completed, now type "end cases" or start a new case.

Coq < suppose it is false.
1 subgoal

*** Declarative Mode ***

A : Prop
B : Prop
x : bool

\[
\text{thesis} := \ B \rightarrow A \lor B
\]

Coq < assume B.
1 subgoal

*** Declarative Mode ***

A : Prop
B : Prop
x : bool
_hyp : B

\[
\text{thesis} := \ A \lor B
\]

Coq < hence B.
Subproof completed, now type "end cases" or start a new case.

Coq < end cases.
Subproof completed, now type "end proof".

11.3.13 Proofs by induction

Proofs by induction are very similar to proofs per cases: they start with per induction on object and proceed with suppose it is pattern and induction hypothesis. The induction hypothesis can be given explicitly or identified by the sub-object m it refers to using thesis for m.

Coq < per induction on n.
1 subgoal
11.3.14 Justifications

Intuitively, justifications are hints for the system to understand how to prove the statements the user types in. In the case of this language justifications are made of two components:

Justification objects: by followed by a comma-separated list of objects that will be used by a selected tactic to prove the statement. This defaults to the empty list (the statement should then be tautological). The * wildcard provides the usual tactics behavior: use all statements in local context. However, this wildcard should be avoided since it reduces the robustness of the script.
Justification tactic: using followed by a Coq tactic that is executed to prove the statement. The default is a solver for (intuitionistic) first-order with equality.

11.4 More details and Formal Semantics

The users looking for more information should have a look at the paper [33]. This paper features a formal semantics of proof state transitions corresponding to the mathematical commands.
Part III

User extensions
Chapter 12

Syntax extensions and interpretation scopes

In this chapter, we introduce advanced commands to modify the way COQ parses and prints objects, i.e. the translations between the concrete and internal representations of terms and commands. The main commands are Notation and Infix which are described in section 12.1. It also happens that the same symbolic notation is expected in different contexts. To achieve this form of overloading, COQ offers a notion of interpretation scope. This is described in Section 12.2.

**Remark:** The commands Grammar, Syntax and Distfix which were present for a while in COQ are no longer available from COQ version 8.0. The underlying AST structure is also no longer available. The functionalities of the command Syntactic Definition are still available, see Section 12.3.

12.1 Notations

12.1.1 Basic notations

A notation is a symbolic abbreviation denoting some term or term pattern.

A typical notation is the use of the infix symbol `\` to denote the logical conjunction (and). Such a notation is declared by

```
Coq < Notation "A \ B" := (and A B).
```

The expression `(and A B)` is the abbreviated term and the string "A \ B" (called a notation) tells how it is symbolically written.

A notation is always surrounded by double quotes (excepted when the abbreviation is a single ident, see 12.3). The notation is composed of tokens separated by spaces. Identifiers in the string (such as A and B) are the parameters of the notation. They must occur at least once each in the denoted term. The other elements of the string (such as `\`) are the symbols.

An identifier can be used as a symbol but it must be surrounded by simple quotes to avoid the confusion with a parameter. Similarly, every symbol of at least 3 characters and starting with a simple quote must be quoted (then it starts by two single quotes). Here is an example.

```
Coq < Notation "'IF' c1 'then' c2 'else' c3" := (IF_then_else c1 c2 c3).
```

A notation binds a syntactic expression to a term. Unless the parser and pretty-printer of COQ already know how to deal with the syntactic expression (see 12.1.7), explicit precedences and associativity rules have to be given.
12.1.2 Precedences and associativity

Mixing different symbolic notations in a same text may cause serious parsing ambiguity. To deal with the ambiguity of notations, COQ uses precedence levels ranging from 0 to 100 (plus one extra level numbered 200) and associativity rules.

Consider for example the new notation

\[ \text{Coq < Notation } "A \lor B" := (\text{or } A \ B). \]

Clearly, an expression such as \( \text{forall } A:\text{Prop}, \ True \lor A \lor A \lor False \) is ambiguous. To tell the COQ parser how to interpret the expression, a priority between the symbols \( \lor \) and \( \land \) has to be given. Assume for instance that we want conjunction to bind more than disjunction. This is expressed by assigning a precedence level to each notation, knowing that a lower level binds more than a higher level. Hence the level for disjunction must be higher than the level for conjunction.

Since connectives are the less tight articulation points of a text, it is reasonable to choose levels not so far from the higher level which is 100, for example 85 for disjunction and 80 for conjunction\(^1\).

Similarly, an associativity is needed to decide whether \( True \lor False \lor False \) defaults to \( True \lor (False \lor False) \) (right associativity) or to \( (True \lor False) \lor False \) (left associativity). We may even consider that the expression is not well-formed and that parentheses are mandatory (this is a “no associativity”\(^2\)). We don’t know of a special convention of the associativity of disjunction and conjunction, let’s apply for instance a right associativity (which is the choice of COQ).

Precedence levels and associativity rules of notations have to be given between parentheses in a list of modifiers that the Notation command understands. Here is how the previous examples refine.

\[ \text{Coq < Notation } "A \lor B" := (\text{and } A \ B) \ (\text{at level 80, right associativity}). \]
\[ \text{Coq < Notation } "A \land B" := (\text{or } A \ B) \ (\text{at level 85, right associativity}). \]

By default, a notation is considered non associative, but the precedence level is mandatory (except for special cases whose level is canonical). The level is either a number or the mention next level whose meaning is obvious. The list of levels already assigned is on Figure 3.1.

12.1.3 Complex notations

Notations can be made from arbitrarily complex symbols. One can for instance define prefix notations.

\[ \text{Coq < Notation } "\neg x" := (\text{not } x) \ (\text{at level 75, right associativity}). \]

One can also define notations for incomplete terms, with the hole expected to be inferred at typing time.

\[ \text{Coq < Notation } "x = y" := (@\text{eq } _ x y) \ (\text{at level 70, no associativity}). \]

One can define closed notations whose both sides are symbols. In this case, the default precedence level for inner subexpression is 200.

\[ \text{Coq < Notation } "(x, y)" := (@\text{pair } _ x y) \ (\text{at level 0}). \]

\(^1\)which are the levels effectively chosen in the current implementation of COQ
\(^2\)COQ accepts notations declared as no associative but the parser on which COQ is built, namely CAMLP4, currently does not implement the no-associativity and replace it by a left associativity; hence it is the same for COQ: no-associativity is in fact left associativity.
One can also define notations for binders.

\[
\text{Coq < Notation } \{ x : A \mid P \} := (\text{sig } A \ (\text{fun } x \Rightarrow P)) \ (\text{at level } 0).
\]

In the last case though, there is a conflict with the notation for type casts. This last notation, as shown by the command \texttt{Print Grammar constr} is at level 100. To avoid \( x : A \) being parsed as a type cast, it is necessary to put \( x \) at a level below 100, typically 99. Hence, a correct definition is

\[
\text{Coq < Notation } \{ x : A \mid P \} := (\text{sig } A \ (\text{fun } x \Rightarrow P)) \ (\text{at level } 0, x \text{ at level } 99).
\]

See the next section for more about factorization.

### 12.1.4 Simple factorization rules

\( \text{Coq} \) extensible parsing is performed by \texttt{Camlp5} which is essentially a LL1 parser. Hence, some care has to be taken not to hide already existing rules by new rules. Some simple left factorization work has to be done. Here is an example.

\[
\text{Coq < Notation } "x < y" := (\text{lt } x \ y) \ (\text{at level } 70).
\]
\[
\text{Coq < Notation } "x < y < z" := (x < y \land y < z) \ (\text{at level } 70).
\]

In order to factorize the left part of the rules, the subexpression referred by \( y \) has to be at the same level in both rules. However the default behavior puts \( y \) at the next level below 70 in the first rule (no associativity is the default), and at the level 200 in the second rule (level 200 is the default for inner expressions). To fix this, we need to force the parsing level of \( y \), as follows.

\[
\text{Coq < Notation } "x < y" := (\text{lt } x \ y) \ (\text{at level } 70).
\]
\[
\text{Coq < Notation } "x < y < z" := (x < y \land y < z) \ (\text{at level } 70, y \text{ at next level}).
\]

For the sake of factorization with \( \text{Coq} \) predefined rules, simple rules have to be observed for notations starting with a symbol: e.g. rules starting with "[" or "(" should be put at level 0. The list of \( \text{Coq} \) predefined notations can be found in Chapter 3.

The command to display the current state of the \( \text{Coq} \) term parser is

\[ \text{Print Grammar constr}.\]

**Variant:**

\[ \text{Print Grammar pattern}.\]

This displays the state of the subparser of patterns (the parser used in the grammar of the \texttt{match with} constructions).
12 Syntax extensions and interpretation scopes

284

12.1.5

Displaying symbolic notations

The command Notation has an effect both on the C OQ parser and on the C OQ printer. For example:
Coq < Check (and True True).
True /\ True
: Prop

However, printing, especially pretty-printing, requires more care than parsing. We may want specific
indentations, line breaks, alignment if on several lines, etc.
The default printing of notations is very rudimentary. For printing a notation, a formatting box is
opened in such a way that if the notation and its arguments cannot fit on a single line, a line break is
inserted before the symbols of the notation and the arguments on the next lines are aligned with the
argument on the first line.
A first, simple control that a user can have on the printing of a notation is the insertion of spaces at
some places of the notation. This is performed by adding extra spaces between the symbols and parameters: each extra space (other than the single space needed to separate the components) is interpreted as
a space to be inserted by the printer. Here is an example showing how to add spaces around the bar of
the notation.
Coq < Notation "{{ x : A | P }}" := (sig (fun x : A => P))
Coq <
(at level 0, x at level 99).
Coq < Check (sig (fun x : nat => x=x)).
{{x : nat | x = x}}
: Set

The second, more powerful control on printing is by using the format modifier. Here is an example
Coq < Notation "’If’ c1 ’then’ c2 ’else’ c3" := (IF_then_else c1 c2 c3)
Coq < (at level 200, right associativity, format
Coq < "’[v
’ ’If’ c1 ’/’ ’[’ ’then’ c2 ’]’ ’/’ ’[’ ’else’ c3 ’]’ ’]’").
Defining ’If’ as keyword

A format is an extension of the string denoting the notation with the possible following elements
delimited by single quotes:
• extra spaces are translated into simple spaces
• tokens of the form ’/ ’ are translated into breaking point, in case a line break occurs, an
indentation of the number of spaces after the “/” is applied (2 spaces in the given example)
• token of the form ’//’ force writing on a new line
• well-bracketed pairs of tokens of the form ’[
’ and ’]’ are translated into printing boxes;
in case a line break occurs, an extra indentation of the number of spaces given after the “[” is
applied (4 spaces in the example)
• well-bracketed pairs of tokens of the form ’[hv
’ and ’]’ are translated into horizontalorelse-vertical printing boxes; if the content of the box does not fit on a single line, then every
breaking point forces a newline and an extra indentation of the number of spaces given after the
“[” is applied at the beginning of each newline (3 spaces in the example)
Coq Reference Manual, V8.2pl2, June 29, 2010


• well-bracketed pairs of tokens of the form ‘[v and ’]’ are translated into vertical printing boxes; every breaking point forces a newline, even if the line is large enough to display the whole content of the box, and an extra indentation of the number of spaces given after the “[” is applied at the beginning of each newline

Thus, for the previous example, we get
Notations do not survive the end of sections. No typing of the denoted expression is performed at definition time. Type-checking is done only at the time of use of the notation.

Coq < Check
Coq < (IF_then_else (IF_then_else True False True) Coq < (IF_then_else True False True))
Coq < (IF_then_else True False True)).

If If True
then False
else True
then If True
then False
else True
else If True
then False
else True
: Prop

Remark: Sometimes, a notation is expected only for the parser. To do so, the option only parsing is allowed in the list of modifiers of Notation.

12.1.6 The Infix command

The Infix command is a shortening for declaring notations of infix symbols. Its syntax is

\text{Infix} \ "symbol" := qualid ( modifier, \ldots , modifier ).

and it is equivalent to

\text{Notation} \ "x symbol y" := (qualid x y) ( modifier, \ldots , modifier ).

where x and y are fresh names distinct from qualid. Here is an example.

Coq < Infix "/\" := and (at level 80, right associativity).

12.1.7 Reserving notations

A given notation may be used in different contexts. Coq expects all uses of the notation to be defined at the same precedence and with the same associativity. To avoid giving the precedence and associativity every time, it is possible to declare a parsing rule in advance without giving its interpretation. Here is an example from the initial state of Coq.

Coq < Reserved Notation "x = y" (at level 70, no associativity).

Reserving a notation is also useful for simultaneously defined an inductive type or a recursive constant and a notation for it.

Remark: The notations mentioned on Figure 3.1 are reserved. Hence their precedence and associativity cannot be changed.

Coq Reference Manual, V8.2pl2, June 29, 2010
12.1.8 Simultaneous definition of terms and notations

Thanks to reserved notations, the inductive, coinductive, recursive and corecursive definitions can benefit of customized notations. To do this, insert a where notation clause after the definition of the (co)inductive type or (co)recursive term (or after the definition of each of them in case of mutual definitions). The exact syntax is given on Figure 12.1. Here are examples:

Coq < Inductive and (A B:Prop) : Prop := conj : A -> B -> A \ B
Coq < where "A \ B" := (and A B).

Coq < Fixpoint plus (n m:nat) {struct n} : nat :=
Coq < match n with
Coq < | O  => m
Coq < | S p => S (p+m)
Coq < end
Coq < where "n + m" := (plus n m).

12.1.9 Displaying informations about notations

To deactivate the printing of all notations, use the command

    Unset Printing Notations.

To reactivate it, use the command

    Set Printing Notations.

The default is to use notations for printing terms wherever possible.

See also: Set Printing All in Section 2.9.

12.1.10 Locating notations

To know to which notations a given symbol belongs to, use the command

    Locate symbol

where symbol is any (composite) symbol surrounded by quotes. To locate a particular notation, use a string where the variables of the notation are replaced by “_”.

Example:

Coq < Locate "exists".
Notation Scope
"'exists' x : t , p" := ex (fun x : t => p)
    : type_scope (default interpretation)
"'exists' x , p" := ex (fun x => p)
    : type_scope (default interpretation)
"'exists' ! x : A , P" := ex (unique (fun x : A => P))
    : type_scope (default interpretation)
"'exists' ! x , P" := ex (unique (fun x => P))
    : type_scope
12.1 Notations

          | Inductive ind_body [decl_notation] with ... with ind_body [decl_notation].
          | CoInductive ind_body [decl_notation] with ... with ind_body [decl_notation].
          | Fixpoint fix_body [decl_notation] with ... with fix_body [decl_notation] .
          | CoFixpoint cofix_body [decl_notation] with ... with cofix_body [decl_notation] .

decl_notation ::= [where string := term [:scope]].

modifiers ::= ident , ... , ident at level natural
             | ident , ... , ident at next level
             | at level natural
             | left associativity
             | right associativity
             | no associativity
             | ident ident
             | ident global
             | ident bigint
             | only parsing
             | format string

Figure 12.1: Syntax of the variants of Notation

(default interpretation)
Coq < Locate "'exists' _ , _.
Notation Scope
"'exists' x , p" := ex (fun x => p)
               : type_scope
       (default interpretation)

See also: Section 6.2.10.

12.1.11 Notations with recursive patterns

An experimental mechanism is provided for declaring elementary notations including recursive patterns.
The basic syntax is

Coq < Notation "[ x ; .. ; y ]" := (cons x .. (cons y nil) ..).

On the right-hand-side, an extra construction of the form .. (f t_1 .. t_n) .. can be used. Notice that .. is part of the COQ syntax while ... is just a meta-notation of this manual to denote a sequence of terms of arbitrary size.

This extra construction enclosed within .., let’s call it t, must be one of the argument of an applicative term of the form (f u_1 .. u_n). The sequences t_1 .. t_n and u_1 .. u_n must coincide everywhere but in two places. In one place, say the terms of indice i, we must have u_i = t. In the other place, say the terms of indice j, both u_j and t_j must be variables, say x and y which are bound by the notation string on the left-hand-side of the declaration. The variables x and y in the string must occur in a substring of the form "x s .. s y" where .. is part of the syntax and s is two times the same sequence of terminal symbols (i.e. symbols which are not variables).
These invariants must be satisfied in order the notation to be correct. The term \( t_i \) is the *terminating* expression of the notation and the pattern \( \{ f \ u_1 \ldots u_{i-1} \ [I] \ u_{i+1} \ldots u_{j-1} \ [E] \ u_{j+1} \ldots u_n \} \) is the *iterating pattern*. The hole \([I]\) is the *iterative* place and the hole \([E]\) is the *enumerating* place. Remark that if \( j < i \), the iterative place comes after the enumerating place accordingly.

The notation parses sequences of tokens such that the subpart "\( s \ldots s \ y \)" parses any number of time (but at least one time) a sequence of expressions separated by the sequence of tokens \( s \). The parsing phase produces a list of expressions which are used to fill in order the holes \([E]\) of the iterating pattern which is nested as many time as the length of the list, the hole \([I]\) being the nesting point. In the innermost occurrence of the nested iterating pattern, the hole \([I]\) is finally filled with the terminating expression.

In the example above, \( f \) is \texttt{cons}, \( n = 3 \) (because \texttt{cons} has a hidden implicit argument!), \( i = 3 \) and \( j = 2 \). The *terminating* expression is \texttt{nil} and the *iterating pattern* is \texttt{cons \ [E] \ [I]}. Finally, the sequence \( s \) is made of the single token "\( ; \)". Here is another example.

Coq < Notation "\( x , y , .. , z \)" := (pair .. (pair x y) .. z) (at level 0).

Notations with recursive patterns can be reserved like standard notations, they can also be declared within interpretation scopes (see section 12.2).

### 12.1.12 Notations and binders

Notations can be defined for binders as in the example:

Coq < Notation "\( \{ x : A \ | \ P \} \)" := (sig (fun x : A => P)) (at level 0).

The binding variables in the left-hand-side that occur as a parameter of the notation naturally bind all their occurrences appearing in their respective scope after instantiation of the parameters of the notation.

Contrastingly, the binding variables that are not a parameter of the notation do not capture the variables of same name that could appear in their scope after instantiation of the notation. E.g., for the notation

Coq < Notation "\'exists\_different\' n" := (exists p:nat, p<>n) (at level 200).

the next command fails because \( p \) does not bind in the instance of \( n \).

Coq < Check (exists\_different\_p).
Coq < Coq < Toplevel input, characters 144-145:
> Check (exists\_different\_p).
> ^
Error: The reference p was not found in the current environment.

**Remark:** Binding variables must not necessarily be parsed using the \texttt{ident} entry. For factorization purposes, they can be said to be parsed at another level (e.g. \( x \) in "\( \{ x : A \ | \ P \} \)" must be parsed at level 99 to be factorized with the notation "\( \{ A \} + \{ B \} \)" for which \( A \) can be any term). However, even if parsed as a term, this term must at the end be effectively a single identifier.
12.2 Interpretation scopes

12.1.13 Summary

Syntax of notations  The different syntactic variants of the command \texttt{Notation} are given on Figure 12.1. The optional \texttt{:scope} is described in the Section 12.2.

Remark: No typing of the denoted expression is performed at definition time. Type-checking is done only at the time of use of the notation.

Remark: Many examples of \texttt{Notation} may be found in the files composing the initial state of COQ (see directory $\texttt{COQLIB/theories/Init}$).

Remark: The notation \texttt{"{ x }"} has a special status in such a way that complex notations of the form \texttt{"x + { y }"} or \texttt{"x * { y }"} can be nested with correct precedences. Especially, every notation involving a pattern of the form \texttt{"{ x }"} is parsed as a notation where the pattern \texttt{"( x )"} has been simply replaced by \texttt{"x"} and the curly brackets are parsed separately. E.g. \texttt{"y + { z }"} is not parsed as a term of the given form but as a term of the form \texttt{"y + z"} where \texttt{z} has been parsed using the rule parsing \texttt{"{ x }"}. Especially, level and precedences for a rule including patterns of the form \texttt{"{ x }"} are relative not to the textual notation but to the notation where the curly brackets have been removed (e.g. the level and the associativity given to some notation, say \texttt{"y \& { z }"} in fact applies to the underlying \texttt{"{ x }"}-free rule which is \texttt{"y \& z"}).

Persistence of notations  Notations do not survive the end of sections. They survive modules unless the command \texttt{Local Notation} is used instead of \texttt{Notation}.

12.2 Interpretation scopes

An interpretation scope is a set of notations for terms with their interpretation. Interpretation scopes provide with a weak, purely syntactical form of notations overloading: a same notation, for instance the infix symbol $+$ can be used to denote distinct definitions of an additive operator. Depending on which interpretation scopes is currently open, the interpretation is different. Interpretation scopes can include an interpretation for numerals and strings. However, this is only made possible at the OBJECTIVE CAML level.

See Figure 12.1 for the syntax of notations including the possibility to declare them in a given scope. Here is a typical example which declares the notation for conjunction in the scope \texttt{type_scope}.

\texttt{Notation "A /\ B" := (and A B) : type_scope.}

Remark: A notation not defined in a scope is called a lonely notation.

12.2.1 Global interpretation rules for notations

At any time, the interpretation of a notation for term is done within a stack of interpretation scopes and lonely notations. In case a notation has several interpretations, the actual interpretation is the one defined by (or in) the more recently declared (or open) lonely notation (or interpretation scope) which defines this notation. Typically if a given notation is defined in some scope \texttt{scope} but has also an interpretation not assigned to a scope, then, if \texttt{scope} is open before the lonely interpretation is declared, then the lonely interpretation is used (and this is the case even if the interpretation of the notation in \texttt{scope} is given after the lonely interpretation: otherwise said, only the order of lonely interpretations and opening of scopes matters, and not the declaration of interpretations within a scope).
The initial state of COQ declares three interpretation scopes and no lonely notations. These scopes, in opening order, are core_scope, type_scope and nat_scope.

The command to add a scope to the interpretation scope stack is

Open Scope scope.

It is also possible to remove a scope from the interpretation scope stack by using the command

Close Scope scope.

Notice that this command does not only cancel the last Open Scope scope but all the invocation of it.

**Remark:** Open Scope and Close Scope do not survive the end of sections where they occur. When defined outside of a section, they are exported to the modules that import the module where they occur.

**Variants:**

1. Local Open Scope scope.
2. Local Close Scope scope.

These variants are not exported to the modules that import the module where they occur, even if outside a section.

### 12.2.2 Local interpretation rules for notations

In addition to the global rules of interpretation of notations, some ways to change the interpretation of subterms are available.

**Local opening of an interpretation scope**

It is possible to locally extend the interpretation scope stack using the syntax (term)%key (or simply term%key for atomic terms), where key is a special identifier called delimiting key and bound to a given scope.

In such a situation, the term term, and all its subterms, are interpreted in the scope stack extended with the scope bound to key.

To bind a delimiting key to a scope, use the command

Delimit Scope scope with ident

**Binding arguments of a constant to an interpretation scope**

It is possible to set in advance that some arguments of a given constant have to be interpreted in a given scope. The command is

Arguments Scope qualid [ opt_scope ... opt_scope ]

where the list is a list made either of _ or of a scope name. Each scope in the list is bound to the corresponding parameter of qualid in order. When interpreting a term, if some of the arguments of qualid are built from a notation, then this notation is interpreted in the scope stack extended by the scopes bound (if any) to these arguments.

**Variants:**
12.2 Interpretation scopes

1. **Global Arguments Scope** \( \text{qualid} [\ \text{opt\_scope} \ldots \text{opt\_scope} ] \)
   
   This behaves like \( \text{Arguments Scope} \ \text{qualid} [\ \text{opt\_scope} \ldots \text{opt\_scope} ] \) but survives when a section is closed instead of stopping working at section closing.

2. **Local Arguments Scope** \( \text{qualid} [\ \text{opt\_scope} \ldots \text{opt\_scope} ] \)
   
   This is a synonym of \( \text{Arguments Scope} \ \text{qualid} [\ \text{opt\_scope} \ldots \text{opt\_scope} ] \): if in a section, the effect of the command stops when the section it belongs to ends.

**See also:** The command to show the scopes bound to the arguments of a function is described in Section 2.

**Binding types of arguments to an interpretation scope**

When an interpretation scope is naturally associated to a type (e.g. the scope of operations on the natural numbers), it may be convenient to bind it to this type. The effect of this is that any argument of a function that syntactically expects a parameter of this type is interpreted using scope. More precisely, it applies only if this argument is built from a notation, and if so, this notation is interpreted in the scope stack extended by this particular scope. It does not apply to the subterms of this notation (unless the interpretation of the notation itself expects arguments of the same type that would trigger the same scope).

More generally, any **class** (see Chapter 17) can be bound to an interpretation scope. The command to do it is

- **Bind Scope** \( \text{scope} \) with **class**

**Example:**

```coq
Coq < Parameter U : Set.
U is assumed

Coq < Bind Scope U_scope with U.

Coq < Parameter Uplus : U -> U -> U.
Uplus is assumed

Coq < Parameter P : forall T:Set, T -> U -> Prop.
P is assumed

Coq < Parameter f : forall T:Set, T -> U.
f is assumed

Coq < Infix "+" := Uplus : U_scope.

Coq < Unset Printing Notations.

Coq < Open Scope nat_scope. (* Define + on the nat as the default for + *)

Coq < Check (fun x y1 y2 z t => P _ (x + t) ((f _ (y1 + y2) + z))).
fun (x y1 y2 : nat) (z : U) (t : nat) =>
P nat (Peano.plus x t) (Uplus (f nat (Peano.plus y1 y2)) z)
  : nat -> nat -> nat -> U -> nat -> Prop
```

**Remark:** The scope **type_scope** has also a local effect on interpretation. See the next section.

**See also:** The command to show the scopes bound to the arguments of a function is described in Section 2.
12 Syntax extensions and interpretation scopes

12.2.3 The type_scope interpretation scope

The scope type_scope has a special status. It is a primitive interpretation scope which is temporarily activated each time a subterm of an expression is expected to be a type. This includes goals and statements, types of binders, domain and codomain of implication, codomain of products, and more generally any type argument of a declared or defined constant.

12.2.4 Interpretation scopes used in the standard library of COQ

We give an overview of the scopes used in the standard library of COQ. For a complete list of notations in each scope, use the commands Print Scopes or Print Scopes scope.

type_scope
This includes infix * for product types and infix + for sum types. It is delimited by key type.

nat_scope
This includes the standard arithmetical operators and relations on type nat. Positive numerals in this scope are mapped to their canonical representent built from O and S. The scope is delimited by key nat.

N_scope
This includes the standard arithmetical operators and relations on type N (binary natural numbers). It is delimited by key N and comes with an interpretation for numerals as closed term of type Z.

Z_scope
This includes the standard arithmetical operators and relations on type Z (binary integer numbers). It is delimited by key Z and comes with an interpretation for numerals as closed term of type Z.

positive_scope
This includes the standard arithmetical operators and relations on type positive (binary strictly positive numbers). It is delimited by key positive and comes with an interpretation for numerals as closed term of type positive.

Q_scope
This includes the standard arithmetical operators and relations on type Q (rational numbers defined as fractions of an integer and a strictly positive integer modulo the equality of the numerator-denominator cross-product). As for numerals, only 0 and 1 have an interpretation in scope Q_scope (their interpretations are \( \frac{0}{1} \) and \( \frac{1}{1} \) respectively).

Qc_scope
This includes the standard arithmetical operators and relations on the type Qc of rational numbers defined as the type of irreducible fractions of an integer and a strictly positive integer.
12.2 Interpretation scopes

This includes the standard arithmetical operators and relations on type $\mathbb{R}$ (axiomatic real numbers). It is delimited by key $\mathbb{R}$ and comes with an interpretation for numerals as term of type $\mathbb{R}$. The interpretation is based on the binary decomposition. The numeral 2 is represented by $1 + 1$. The interpretation $\phi(n)$ of an odd positive numerals greater than $n$ than 3 is $1 + (1+1) \times \phi((n - 1)/2)$. The interpretation $\phi(n)$ of an even positive numerals greater $n$ than 4 is $(1+1) \times \phi(n/2)$. Negative numerals are represented as the opposite of the interpretation of their absolute value. E.g. the syntactic object $-11$ is interpreted as $-(1+(1+1)\times((1+1)\times(1+(1+1))))$ where the unit 1 and all the operations are those of $\mathbb{R}$.

bool_scope
This includes notations for the boolean operators. It is delimited by key bool.

list_scope
This includes notations for the list operators. It is delimited by key list.

core_scope
This includes the notation for pairs. It is delimited by key core.

string_scope
This includes notation for strings as elements of the type string. Special characters and escaping follow COQ conventions on strings (see Section 1.1). Especially, there is no convention to visualize non printable characters of a string. The file String.v shows an example that contains quotes, a newline and a beep (i.e. the ascii character of code 7).

char_scope
This includes interpretation for all strings of the form "c" where c is an ascii character, or of the form "nnn" where nnn is a three-digits number (possibly with leading 0's), or of the form "\". Their respective denotations are the ascii code of c, the decimal ascii code nnn, or the ascii code of the character " (i.e. the ascii code 34), all of them being represented in the type ascii.

12.2.5 Displaying informations about scopes

Print Visibility
This displays the current stack of notations in scopes and lonely notations that is used to interpret a notation. The top of the stack is displayed last. Notations in scopes whose interpretation is hidden by the same notation in a more recently open scope are not displayed. Hence each notation is displayed only once.

Variant:
Print Visibility scope

This displays the current stack of notations in scopes and lonely notations assuming that scope is pushed on top of the stack. This is useful to know how a subterm locally occurring in the scope of scope is interpreted.
Print Scope `scope`

This displays all the notations defined in interpretation scope `scope`. It also displays the delimiting key if any and the class to which the scope is bound, if any.

Print Scopes

This displays all the notations, delimiting keys and corresponding class of all the existing interpretation scopes. It also displays the lonely notations.

### 12.3 Abbreviations

An abbreviation is a name, possibly applied to arguments, that denotes a (presumably) more complex expression. Here are examples:

Coq < Notation Nlist := (list nat).
Coq < Check 1 :: 2 :: 3 :: nil.
   [1; 2; 3] : Nlist
Coq < Notation reflexive R := (forall x, R x x).
Coq < Check forall A:Prop, A <-> A.
   reflexive iff : Prop
Coq < Check reflexive iff.
   reflexive iff : Prop

An abbreviation expects no precedence nor associativity, since it follows the usual syntax of application. Abbreviations are used as much as possible by the Coq printers unless the modifier (only parsing) is given.

Abbreviations are bound to an absolute name as an ordinary definition is, and they can be referred by qualified names too.

Abbreviations are syntactic in the sense that they are bound to expressions which are not typed at the time of the definition of the abbreviation but at the time it is used. Especially, abbreviations can be bound to terms with holes (i.e. with “_”). The general syntax for abbreviations is

```
[Local] Notation ident [ident ident ... ident ident] := term [(only parsing)].
```

Example:

Coq < Definition explicit_id (A:Set) (a:A) := a.
   explicit_id is defined
Coq < Notation id := (explicit_id _).
Coq < Check (id 0).
   id 0 : nat

Abbreviations do not survive the end of sections. No typing of the denoted expression is performed at definition time. Type-checking is done only at the time of use of the abbreviation.
12.4 Tactic Notations

Tactic notations allow to customize the syntax of the tactics of the tactic language. Tactic notations obey the following syntax:

\[
\text{sentence} ::= \text{Tactic Notation} [\text{tactic_level}] \text{ production_item} \ldots \text{ production_item} := \text{ tactic} .
\]

\[
\text{production_item} ::= \text{ string} | \text{ tactic_argument_type} (\text{ ident})
\]

\[
\text{tactic_level} ::= (\text{ at level } \text{ natural})
\]

\[
\text{tactic_argument_type} ::= \text{ ident} | \text{ simple_intropattern} | \text{ reference} | \text{ hyp} | \text{ hyp_list} | \text{ ne_hyp_list} | \text{ constr} | \text{ constr_list} | \text{ ne_constr_list} | \text{ integer} | \text{ integer_list} | \text{ ne_integer_list} | \text{ int_or_var} | \text{ int_or_var_list} | \text{ ne_int_or_var_list} | \text{ tactic} | \text{ tactic}_n \quad (\text{ for } 0 \leq n \leq 5)
\]

A tactic notation \text{Tactic Notation} \text{ tactic_level} \text{ [production_item} \ldots \text{ production_item}] := \text{ tactic} extends the parser and pretty-printer of tactics with a new rule made of the list of production items. It then evaluates into the tactic expression \text{ tactic}. For simple tactics, it is recommended to use a terminal symbol, i.e., a \text{ string}, for the first production item. The tactic level indicates the parsing precedence of the tactic notation. This information is particularly relevant for notations of tacticals. Levels 0 to 5 are available (default is 0). To know the parsing precedences of the existing tacticals, use the command \text{ Print Grammar tactic}.

Each type of tactic argument has a specific semantic regarding how it is parsed and how it is interpreted. The semantic is described in the following table. The last command gives examples of tactics which use the corresponding kind of argument.

<table>
<thead>
<tr>
<th>Tactic argument type</th>
<th>parsed as</th>
<th>interpreted as</th>
<th>as in tactic</th>
</tr>
</thead>
<tbody>
<tr>
<td>ident</td>
<td>identifier</td>
<td>a user-given name</td>
<td>intro</td>
</tr>
<tr>
<td>simple_intropattern</td>
<td>intro_pattern</td>
<td>an intro_pattern</td>
<td>intros</td>
</tr>
<tr>
<td>hyp</td>
<td>identifier</td>
<td>an hypothesis defined in context</td>
<td>clear</td>
</tr>
<tr>
<td>reference</td>
<td>qualified identifier</td>
<td>a global reference of term</td>
<td>unfold</td>
</tr>
<tr>
<td>constr</td>
<td>term</td>
<td>a term</td>
<td>exact</td>
</tr>
<tr>
<td>integer</td>
<td>integer</td>
<td>an integer</td>
<td>do</td>
</tr>
<tr>
<td>int_or_var</td>
<td>identifier or integer</td>
<td>an integer</td>
<td></td>
</tr>
<tr>
<td>tactic</td>
<td>tactic at level 5</td>
<td>a tactic</td>
<td></td>
</tr>
<tr>
<td>tactic_n</td>
<td>tactic at level (n)</td>
<td>a tactic</td>
<td></td>
</tr>
<tr>
<td>entry_list</td>
<td>list of (entry)</td>
<td>a list of how (entry) is interpreted</td>
<td></td>
</tr>
<tr>
<td>ne_entry_list</td>
<td>non-empty list of (entry)</td>
<td>a list of how (entry) is interpreted</td>
<td></td>
</tr>
</tbody>
</table>

**Remark:** In order to be bound in tactic definitions, each syntactic entry for argument type must include the case of simple \(\text{ L}_{\text{ tac}}\) identifier as part of what it parses. This is naturally the case for \text{ident}, \text{simple_intropattern}, \text{reference}, \text{constr}, ... but not for \text{integer}. This is the reason for introducing a special entry \text{ int_or_var} which evaluates to integers only but which syntactically includes identifiers in order to be usable in tactic definitions.

---

3Tactic notations are just a simplification of the Grammar tactic simple_tactic command that existed in versions prior to version 8.0.
**Remark:** The `entry_list` and `ne_entry_list` entries can be used in primitive tactics or in other notations at places where a list of the underlying entry can be used: `entry` is either `constr`, `hyp`, `integer` or `int_or_var`. 
Part IV

Practical tools
Chapter 13

The COQ commands

There are three COQ commands:

- **coqtop**: The COQ toplevel (interactive mode);
- **coqc**: The COQ compiler (batch compilation);
- **coqchk**: The COQ checker (validation of compiled libraries).

The options are (basically) the same for the first two commands, and roughly described below. You can also look at the **man** pages of **coqtop** and **coqc** for more details.

### 13.1 Interactive use (**coqtop**)

In the interactive mode, also known as the COQ toplevel, the user can develop his theories and proofs step by step. The COQ toplevel is run by the command **coqtop**.

They are two different binary images of COQ: the byte-code one and the native-code one (if Objective Caml provides a native-code compiler for your platform, which is supposed in the following). When invoking **coqtop** or **coqc**, the native-code version of the system is used. The command-line options **-byte** and **-opt** explicitly select the byte-code and the native-code versions, respectively.

The byte-code toplevel is based on a Caml toplevel (to allow the dynamic link of tactics). You can switch to the Caml toplevel with the command **Drop.**, and come back to the COQ toplevel with the command **Toplevel.loop();**.

### 13.2 Batch compilation (**coqc**)

The **coqc** command takes a name **file** as argument. Then it looks for a vernacular file named **file.v**, and tries to compile it into a **file.vo** file (See 6.4).

**Warning**: The name **file** must be a regular COQ identifier, as defined in the Section 1.1. It must only contain letters, digits or underscores (_). Thus it can be **/bar/foo/toto.v** but cannot be **/bar/foo/to-to.v**.

Notice that the **-byte** and **-opt** options are still available with **coqc** and allow you to select the byte-code or native-code versions of the system.
13.3 Resource file

When Coq is launched, with either coqtop or coqc, the resource file $HOME/.coqrc.7.0 is loaded, where $HOME is the home directory of the user. If this file is not found, then the file $HOME/.coqrc is searched. You can also specify an arbitrary name for the resource file (see option -init-file below), or the name of another user to load the resource file of someone else (see option -user).

This file may contain, for instance, Add LoadPath commands to add directories to the load path of Coq. It is possible to skip the loading of the resource file with the option -q.

13.4 Environment variables

There are three environment variables used by the Coq system. $COQBIN for the directory where the binaries are, $COQLIB for the directory where the standard library is, and $COQTOP for the directory of the sources. The latter is useful only for developers that are writing their own tactics and are using coq_makefile (see 14.3). If $COQBIN or $COQLIB are not defined, Coq will use the default values (defined at installation time). So these variables are useful only if you move the Coq binaries and library after installation.

13.5 Options

The following command-line options are recognized by the commands coqc and coqtop, unless stated otherwise:

-byte

Run the byte-code version of Coq.

-opt

Run the native-code version of Coq.

-I directory, --include directory

Add physical path directory to the list of directories where to look for a file and bind it to the empty logical directory. The subdirectory structure of directory is recursively available from Coq using absolute names (see Section 2.6.2).

-I directory -as dirpath

Add physical path directory to the list of directories where to look for a file and bind it to the logical directory dirpath. The subdirectory structure of directory is recursively available from Coq using absolute names extending the dirpath prefix.

See also: Add LoadPath in Section 6.5.3 and logical paths in Section 2.6.1.

-R directory dirpath, -R directory -as dirpath

Do as -I directory -as dirpath but make the subdirectory structure of directory recursively visible so that the recursive contents of physical directory is available from Coq using short or partially qualified names.

See also: Add Rec LoadPath in Section 6.5.4 and logical paths in Section 2.6.1.
13.5 Options

- **top** *dirpath*
  This sets the toplevel module name to *dirpath* instead of Top. Not valid for coqc.

- **notop** *dirpath*
  This sets the toplevel module name to the empty logical dirpath. Not valid for coqc.

- **exclude-dir** *subdirectory*
  This tells to exclude any subdirectory named *subdirectory* while processing option -R. Without this option only the conventional version control management subdirectories named CVS and _darcs are excluded.

- **is** *file*, **-inputstate** *file*
  Cause Coq to use the state put in the file *file* as its input state. The default state is *initial.coq*. Mainly useful to build the standard input state.

- **outputstate** *file*
  Cause Coq to dump its state to file *file*.coq just after finishing parsing and evaluating all the arguments from the command line.

- **nois**
  Cause Coq to begin with an empty state. Mainly useful to build the standard input state.

- **init-file** *file*
  Take *file* as the resource file.

- **-q**
  Cause Coq not to load the resource file.

- **user** *username*
  Take resource file of user *username* (that is ~/username/.coqrc.7.0) instead of yours.

- **load-ml-source** *file*
  Load the Caml source file *file*.

- **load-ml-object** *file*
  Load the Caml object file *file*.

- **l** *file*, **-load-vernac-source** *file*
  Load Coq file *file*.v

- **lv** *file*, **-load-vernac-source-verbose** *file*
  Load Coq file *file*.v with a copy of the contents of the file on standard input.

- **load-vernac-object** *file*
  Load Coq compiled file *file*.vo

- **require** *file*
  Load Coq compiled file *file*.vo and import it (Require file).
-compile file

This compiles file file.v into file.vo. This option implies options -batch and -silent. It is only available for coqtop.

-compile-verbose file

This compiles file file.v into file.vo with a copy of the contents of the file on standard input. This option implies options -batch and -silent. It is only available for coqtop.

-verbose

This option is only for coqc. It tells to compile the file with a copy of its contents on standard input.

-batch

Batch mode : exit just after arguments parsing. This option is only used by coqc.

-xml

This option is for use with coqc. It tells Coq to export on the standard output the content of the compiled file into XML format.

-quality Improve the legibility of the proof terms produced by some tactics.

-emacs

Tells Coq it is executed under Emacs.

-impredicative-set

Change the logical theory of Coq by declaring the sort Set impredicative; warning: this is known to be inconsistent with some standard axioms of classical mathematics such as the functional axiom of choice or the principle of description

-dump-glob file

This dumps references for global names in file file (to be used by coqdoc, see 14.4)

-dont-load-proofs

This avoids loading in memory the proofs of opaque theorems resulting in a smaller memory requirement and faster compilation; warning: this invalidates some features such as the extraction tool.

-vm

This activates the use of the bytecode-based conversion algorithm for the current session (see Section 6.9.4).

-image file

This option sets the binary image to be used to be file instead of the standard one. Not of general use.

-bindir directory

Set for coqc the directory containing Coq binaries. It is equivalent to do export COQBIN=directory before lauching coqc.
13.6 Compiled libraries checker (*coqchk*)

The *coqchk* command takes a list of library paths as argument. The corresponding compiled libraries (.vo files) are searched in the path, recursively processing the libraries they depend on. The content of all these libraries is then type-checked. The effect of *coqchk* is only to return with normal exit code in case of success, and with positive exit code if an error has been found. Error messages are not deemed to help the user understand what is wrong. In the current version, it does not modify the compiled libraries to mark them as successfully checked.

Note that non-logical information is not checked. By logical information, we mean the type and optional body associated to names. It excludes for instance anything related to the concrete syntax of objects (customized syntax rules, association between short and long names), implicit arguments, etc.

This tool can be used for several purposes. One is to check that a compiled library provided by a third-party has not been forged and that loading it cannot introduce inconsistencies. Another point is to get an even higher level of security. Since *coqtop* can be extended with custom tactics, possibly ill-typed code, it cannot be guaranteed that the produced compiled libraries are correct. *coqchk* is a standalone verifier, and thus it cannot be tainted by such malicious code.

Command-line options `-I`, `-R`, `-where` and `-impredicative-set` are supported by *coqchk* and have the same meaning as for *coqtop*. Extra options are:

- `norec module`
  Check *module* but do not force check of its dependencies.

- `admit module`
  Do not check *module* and any of its dependencies, unless explicitly required.

- `-o`
  At exit, print a summary about the context. List the names of all assumptions and variables (constants without body).

- `silent`
  Do not write progress information in standard output.

Environment variable `$COQLIB` can be set to override the location of the standard library.

The algorithm for deciding which modules are checked or admitted is the following: assuming that *coqchk* is called with argument *M*, option `norec N`, and `admit A`. Let us write $S$ the set of reflexive transitive dependencies of set $S$. Then:

---

1Ill-formed non-logical information might for instance bind `Coq.Init.Logic.True` to short name `False`, so Apparently `False` is inhabited, but using fully qualified names, `Coq.Init.Logic.False` will always refer to the absurd proposition, what we guarantee is that there is no proof of this latter constant.

Coq Reference Manual, V8.2pl2, June 29, 2010
• Modules $C = M \setminus A \cup M \cup N$ are loaded and type-checked before being added to the context.

• And $M \cup N \setminus C$ is the set of modules that are loaded and added to the context without type-checking. Basic integrity checks (checksums) are nonetheless performed.

As a rule of thumb, the `-admit` can be used to tell that some libraries have already been checked. So `coqchk A B` can be split in `coqchk A && coqchk B -admit A` without type-checking any definition twice. Of course, the latter is slightly slower since it makes more disk access. It is also less secure since an attacker might have replaced the compiled library $A$ after it has been read by the first command, but before it has been read by the second command.
Chapter 14

Utilities

The distribution provides utilities to simplify some tedious works beside proof development, tactics writing or documentation.

14.1 Building a toplevel extended with user tactics

The native-code version of COQ cannot dynamically load user tactics using Objective Caml code. It is possible to build a toplevel of COQ, with Objective Caml code statically linked, with the tool coqmktop.

For example, one can build a native-code COQ toplevel extended with a tactic which source is in tactic.ml with the command

```%
coqmktop -opt -o mytop.out tactic.cmx
```

where tactic.ml has been compiled with the native-code compiler ocamlopt. This command generates an executable called mytop.out. To use this executable to compile your COQ files, use coqc -image mytop.out.

A basic example is the native-code version of COQ (coqtop.opt), which can be generated by coqmktop -opt -o coqopt.opt.

Application: how to use the Objective Caml debugger with Coq. One useful application of coqmktop is to build a COQ toplevel in order to debug your tactics with the Objective Caml debugger. You need to have configured and compiled COQ for debugging (see the file INSTALL included in the distribution). Then, you must compile the Caml modules of your tactic with the option -g (with the bytecode compiler) and build a stand-alone bytecode toplevel with the following command:

```%
coqmktop -g -o coq-debug <your.cmo files>
```

To launch the OBJECTIVE CAML debugger with the image you need to execute it in an environment which correctly sets the COQLIB variable. Moreover, you have to indicate the directories in which ocamldebug should search for Caml modules.

A possible solution is to use a wrapper around ocamldebug which detects the executables containing the word coq. In this case, the debugger is called with the required additional arguments. In other cases, the debugger is simply called without additional arguments. Such a wrapper can be found in the dev/ subdirectory of the sources.
14.2 Modules dependencies

In order to compute modules dependencies (so to use make), COQ comes with an appropriate tool, coqdep.

coqdep computes inter-module dependencies for COQ and OBJECTIVE CAML programs, and prints the dependencies on the standard output in a format readable by make. When a directory is given as argument, it is recursively looked at.

Dependencies of COQ modules are computed by looking at Require commands (Require, Require Export, Require Import, Require Implementation), but also at the command Declare ML Module.

Dependencies of OBJECTIVE CAML modules are computed by looking at open commands and the dot notation module.value. However, this is done approximatively and you are advised to use ocamldep instead for the OBJECTIVE CAML modules dependencies.

See the man page of coqdep for more details and options.

14.3 Creating a Makefile for COQ modules

When a proof development becomes large and is split into several files, it becomes crucial to use a tool like make to compile COQ modules.

The writing of a generic and complete Makefile may be a tedious work and that’s why COQ provides a tool to automate its creation, coq_makefile. Given the files to compile, the command coq_makefile prints a Makefile on the standard output. So one has just to run the command:

% coq_makefile file1.v ... file_n.v > Makefile

The resulted Makefile has a target depend which computes the dependencies and puts them in a separate file .depend, which is included by the Makefile. Therefore, you should create such a file before the first invocation of make. You can for instance use the command

% touch .depend

Then, to initialize or update the modules dependencies, type in:

% make depend

There is a target all to compile all the files file_1 ... file_n, and a generic target to produce a .vo file from the corresponding .v file (so you can do make file_vo to compile the file file_v).

cq_makefile can also handle the case of ML files and subdirectories. For more options type

% coq_makefile -help

Warning: To compile a project containing OBJECTIVE CAML files you must keep the sources of COQ somewhere and have an environment variable named COQTOP that points to that directory.
### 14.4 Documenting COQ files with coqdoc

**coqdoc** is a documentation tool for the proof assistant COQ, similar to **javadoc** or **ocamldoc**. The task of **coqdoc** is

1. to produce a nice **latex** and/or HTML document from the COQ sources, readable for a human and not only for the proof assistant;

2. to help the user navigating in his own (or third-party) sources.

#### 14.4.1 Principles

Documentation is inserted into COQ files as *special comments*. Thus your files will compile as usual, whether you use **coqdoc** or not. **coqdoc** presupposes that the given COQ files are well-formed (at least lexically). Documentation starts with (**, followed by a space, and ends with the pending **). The documentation format is inspired by Todd A. Coram’s *Almost Free Text (AFT)* tool: it is mainly ASCII text with some syntax-light controls, described below. **coqdoc** is robust: it shouldn’t fail, whatever the input is. But remember: “garbage in, garbage out”.

**COQ material inside documentation.** COQ material is quoted between the delimiters [ and ]. Square brackets may be nested, the inner ones being understood as being part of the quoted code (thus you can quote a term like \( x : T \) by writing \([x:T]u\)). Inside quotations, the code is pretty-printed in the same way as it is in code parts.

Pre-formatted vernacular is enclosed by \([ [ \) and \( ] ]\). The former must be followed by a newline and the latter must follow a newline.

**Pretty-printing.** **coqdoc** uses different faces for identifiers and keywords. The pretty-printing of COQ tokens (identifiers or symbols) can be controlled using one of the following commands:

(*** printing token \%...\text{\LaTeX}...\% #...\text{HTML}...# **)

or

(*** printing token $...\text{\LaTeX\ math}...$ #...\text{HTML}...# **)

It gives the **latex** and HTML texts to be produced for the given COQ token. One of the **latex** or HTML text may be ommitted, causing the default pretty-printing to be used for this token.

The printing for one token can be removed with

(*** remove printing token **)

Initially, the pretty-printing table contains the following mapping:

<table>
<thead>
<tr>
<th>Symbol</th>
<th>LATEX</th>
<th>HTML</th>
</tr>
</thead>
<tbody>
<tr>
<td>-&gt;</td>
<td>(\rightarrow)</td>
<td>(\Rightarrow)</td>
</tr>
<tr>
<td>&lt;=</td>
<td>(\leq)</td>
<td>(\Rightarrow)</td>
</tr>
<tr>
<td>==</td>
<td>(\equiv)</td>
<td>(\Rightarrow)</td>
</tr>
<tr>
<td>&lt;&gt;</td>
<td>(\neq)</td>
<td>(\Rightarrow)</td>
</tr>
<tr>
<td>|</td>
<td>(\lor)</td>
<td>(\Rightarrow)</td>
</tr>
<tr>
<td>\</td>
<td>(\land)</td>
<td>(\Rightarrow)</td>
</tr>
</tbody>
</table>

Any of these can be overwritten or suppressed using the **printing** commands.

Important note: the recognition of tokens is done by a (ocaml)lex automaton and thus applies the longest-match rule. For instance, \(-\sim\) is recognized as a single token, where COQ sees two tokens. It is the responsibility of the user to insert space between tokens or to give pretty-printing rules for the possible combinations, e.g.

(*** printing \(-\sim\) %\text{ensuremath}{\\rightarrow\\not\sim}\% **)
Sections. Sections are introduced by 1 to 4 leading stars (i.e. at the beginning of the line) followed by a space. One star is a section, two stars a sub-section, etc. The section title is given on the remaining of the line. Example:

(** * Well-founded relations

       In this section, we introduce... *)

Lists. List items are introduced by 1 to 4 leading dashes. Deepness of the list is indicated by the number of dashes. List ends with a blank line. Example:

       This module defines
       - the predecessor [pred]
       - the addition [plus]
       - order relations:
         -- less or equal [le]
         -- less [lt]

Rules. More than 4 leading dashes produce an horizontal rule.

Escapings to \LaTeX\ and HTML. Pure \LaTeX\ or HTML material can be inserted using the following escape sequences:

- \$...LaTeX stuff...\$ inserts some \LaTeX\ material in math mode. Simply discarded in HTML output.
- %...LaTeX stuff...% inserts some \LaTeX\ material. Simply discarded in HTML output.
- #...HTML stuff...# inserts some HTML material. Simply discarded in \LaTeX\ output.

Note: to simply output the characters $, % and # and escaping their escaping role, these characters must be doubled.

Verbatim. Verbatim material is introduced by a leading << and closed by >> at the beginning of a line. Example:

Here is the corresponding caml code:
<<
  let rec fact n =
  if n <= 1 then 1 else n * fact (n-1)
>>

Hyperlinks. Hyperlinks can be inserted into the HTML output, so that any identifier is linked to the place of its definition.

In order to get hyperlinks you need to first compile your COQ file using coqc --dump-glob file; this appends COQ names resolutions done during the compilation to file file. Take care of erasing this file, if any, when starting the whole compilation process.

Then invoke coqdoc --glob-from file to tell coqdoc to look for name resolutions into the file file.
Identifiers from the COQ standard library are linked to the COQ web site at http://coq.inria. fr/library/. This behavior can be changed using command line options --no-externals and --coqlib; see below.

Hiding / Showing parts of the source. Some parts of the source can be hidden using command line options -g and -l (see below), or using such comments:

(* begin hide *)
some Coq material
(* end hide *)

Conversely, some parts of the source which would be hidden can be shown using such comments:

(* begin show *)
some Coq material
(* end show *)

The latter cannot be used around some inner parts of a proof, but can be used around a whole proof.

14.4.2 Usage

cqdoc is invoked on a shell command line as follows:

cqdoc < options and files >

Any command line argument which is not an option is considered to be a file (even if it starts with a -). COQ files are identified by the suffixes .v and .g and LATEX files by the suffix .tex.

HTML output

This is the default output. One HTML file is created for each COQ file given on the command line, together with a file index.html (unless option -no-index is passed). The HTML pages use a style sheet named style.css. Such a file is distributed with cqdoc.

LATEX output

A single LATEX file is created, on standard output. It can be redirected to a file with option -o. The order of files on the command line is kept in the final document. LATEX files given on the command line are copied 'as is' in the final document. DVI and PostScript can be produced directly with the options -dvi and -ps respectively.

TEXmacs output

To translate the input files to TEXmacs format, to be used by the TEXmacs Coq interface (see http://www-sop.inria.fr/lemme/Philippe.Audebaud/tmcoq/).

Command line options

Overall options

--html

Select a HTML output.
--latex
Select a \LaTeX output.

--dvi
Select a DVI output.

--ps
Select a PostScript output.

--texmacs
Select a \TeXmacs output.

--stdout
Write output to stdout.

-o file, --output file
Redirect the output into the file ‘file’ (meaningless with --html).

-d dir, --directory dir
Output files into directory ‘dir’ instead of current directory (option --d does not change the file-name specified with option --o, if any).

-s, --short
Do not insert titles for the files. The default behavior is to insert a title like “Library Foo” for each file.

-t string, --title string
Set the document title.

--body-only
Suppress the header and trailer of the final document. Thus, you can insert the resulting document into a larger one.

-p string, --preamble string
Insert some material in the \LaTeX preamble, right before \begin{document} (meaningless with --html).

--vernac-file file, --tex-file file
Considers the file ‘file’ respectively as a .v (or .g) file or a .tex file.

--files-from file
Read file names to process in file ‘file’ as if they were given on the command line. Useful for program sources splitted in several directories.

-q, --quiet
Be quiet. Do not print anything except errors.
14.4 Documenting Coq files with coqdoc

-\texttt{h, --help}
  
  Give a short summary of the options and exit.

-\texttt{v, --version}
  
  Print the version and exit.

\textbf{Index options}

Default behavior is to build an index, for the HTML output only, into index.html.

-\texttt{--no-index}
  
  Do not output the index.

-\texttt{--multi-index}
  
  Generate one page for each category and each letter in the index, together with a top page index.html.

\textbf{Table of contents option}

-\texttt{toc, --table-of-contents}
  
  Insert a table of contents. For a \LaTeX{} output, it inserts a \texttt{\tableofcontents} at the beginning of the document. For a HTML output, it builds a table of contents into toc.html.

\textbf{Hyperlinks options}

-\texttt{--glob-from file}
  
  Make references using Coq globalizations from file \texttt{file}. (Such globalizations are obtained with Coq option \texttt{-dump-glob}).

-\texttt{--no-externals}
  
  Do not insert links to the Coq standard library.

-\texttt{--coqlib url}
  
  Set base URL for the Coq standard library (default is http://coq.inria.fr/library/).

-\texttt{R dir coqdir}
  
  Map physical directory \texttt{dir} to Coq logical directory \texttt{coqdir} (similarly to Coq option \texttt{-R}).
  
  Note: option \texttt{-R} only has effect on the files \texttt{following} it on the command line, so you will probably need to put this option first.

\textbf{Contents options}

-\texttt{g, --gallina}
  
  Do not print proofs.

-\texttt{l, --light}
  
  Light mode. Suppress proofs (as with \texttt{-g}) and the following commands:
The behavior of options -g and -l can be locally overridden using the (* begin show *) ... (* end show *) environment (see above).

Language options
Default behavior is to assume ASCII 7 bits input files.

-latin1, --latin1
Select ISO-8859-1 input files. It is equivalent to -inputenc latin1 -charset iso-8859-1.

-utf8, --utf8
Select UTF-8 (Unicode) input files. It is equivalent to -inputenc utf8 -charset utf-8. \LaTeX UTF-8 support can be found at http://www.ctan.org/tex-archive/macros/latex/contrib/supported/unicode/.

--inputenc string
Give a \LaTeX input encoding, as an option to \LaTeX package inputenc.

--charset string
Specify the HTML character set, to be inserted in the HTML header.

14.4.3 The coqdoc \LaTeX style file
In case you choose to produce a document without the default \LaTeX preamble (by using option --no-preamble), then you must insert into your own preamble the command

\usepackage{coqdoc}

Then you may alter the rendering of the document by redefining some macros:

\texttt{coqdockw, coqdocid}

The one-argument macros for typesetting keywords and identifiers. Defaults are sans-serif for keywords and italic for identifiers.

For example, if you would like a slanted font for keywords, you may insert

\renewcommand{\coqdockw}[1]{\textsl{#1}}

anywhere between \usepackage{coqdoc} and \begin{document}. 
coqdocmodule

One-argument macro for typesetting the title of a .v file. Default is
\newcommand{\coqdocmodule}[1]{\section*{Module #1}}

and you may redefine it using \renewcommand.

14.5 Exporting COQ theories to XML

This section describes the exportation of COQ theories to XML that has been contributed by Claudio Sacerdoti Coen. Currently, the main applications are the rendering and searching tool developed within the HELM\textsuperscript{1} and MoWGLI\textsuperscript{2} projects mainly at the University of Bologna and partly at INRIA-Sophia Antipolis.

14.5.1 Practical use of the XML exportation tool

The basic way to export the logical content of a file into XML format is to use \texttt{coqc} with option \texttt{-xml}. When the \texttt{-xml} flag is set, every definition or declaration is immediately exported to XML once concluded. The system environment variable \texttt{COQ\_XML\_LIBRARY\_ROOT} must be previously set to a directory in which the logical structure of the exported objects is reflected.

For Makefile files generated by \texttt{coq\_makefile} (see section 14.3), it is sufficient to compile the files using

\begin{verbatim}
make COQ\_XML=-xml
\end{verbatim}

To export a development to XML, the suggested procedure is then:

1. add to your own contribution a valid Make file and use \texttt{coq\_makefile} to generate the Make file from the Make file.

   \textbf{Warning:} Since logical names are used to structure the XML hierarchy, always add to the Make file at least one "-R" option to map physical file names to logical module paths.

2. set the \texttt{COQ\_XML\_LIBRARY\_ROOT} environment variable to the directory where the XML file hierarchy must be physically rooted.

3. compile your contribution with "make COQ\_XML=-xml"

\textbf{Remark:} In case the system variable \texttt{COQ\_XML\_LIBRARY\_ROOT} is not set, the output is done on the standard output. Also, the files are compressed using gzip after creation. This is to save disk space since the XML format is very verbose.

14.5.2 Reflection of the logical structure into the file system

For each COQ logical object, several independent files associated to this object are created. The structure of the long name of the object is reflected in the directory structure of the file system. E.g. an object of long name \texttt{ident1/\ldots/identn/ident} is exported to files in the subdirectory \texttt{ident1/\ldots/identn} of the directory bound to the environment variable \texttt{COQ\_XML\_LIBRARY\_ROOT}.

---
\textsuperscript{1}Hypertextual Electronic Library of Mathematics
\textsuperscript{2}Mathematics on the Web, Get it by Logic and Interfaces
14.5.3 What is exported?

The XML exportation tool exports the logical content of COQ theories. This covers global definitions (including lemmas, theorems, ...), global assumptions (parameters and axioms), local assumptions or definitions, and inductive definitions.

Vernacular files are exported to .theory.xml files. Comments are pre-processed with coqdoc (see section 14.4). Especially, they have to be enclosed within (** and *) to be exported.

For each inductive definition of name ident_1,...,ident_n, a file named ident.ind.xml is created in the subdirectory ident_1/.../ident_n of the xml library root directory. It contains the arities and constructors of the type. For mutual inductive definitions, the file is named after the name of the first inductive type of the block.

For each global definition of base name ident_1,...,ident_n, files named ident.con.body.xml and ident.con.xml are created in the subdirectory ident_1/.../ident_n. They respectively contain the body and the type of the definition.

For each global assumption of base name ident_1,...,ident_n, a file named ident.con.xml is created in the subdirectory ident_1/.../ident_n. It contains the type of the global assumption.

For each local assumption or definition of base name ident located in sections ident'_1,...,ident'_p of the module ident_1,ident_2,...,ident_n, ident, a file named ident.var.xml is created in the subdirectory ident_1/.../ident_n/ident'_1/.../ident'_p. It contains its type and, if a definition, its body.

In order to do proof-rendering (for example in natural language), some redundant typing information is required, i.e. the type of at least some of the subterms of the bodies and types of the CIC objects. These types are called inner types and are exported to files of suffix .types.xml by the exportation tool.

14.5.4 Inner types

The type of a subterm of a construction is called an inner type if it respects the following conditions.

1. Its sort is Prop^3.

2. It is not a type cast nor an atomic term (variable, constructor or constant).

3. If it’s root is an abstraction, then the root’s parent node is not an abstraction, i.e. only the type of the outer abstraction of a block of nested abstractions is printed.

The rationale for the 3rd condition is that the type of the inner abstractions could be easily computed starting from the type of the outer ones; moreover, the types of the inner abstractions requires a lot of disk/memory space: removing the 3rd condition leads to XML file that are two times as big as the ones exported applying the 3rd condition.

14.5.5 Interactive exportation commands

There are also commands to be used interactively in coqtop.

---

^3 or CProp which is the "sort"-like definition used in C-CoRN (see http://vacuumcleaner.cs.kun.nl/c-corn) to type computationally relevant predicative propositions.
14.5 Exporting CoQ theories to XML

Print XML \textit{qualid}

If the variable \texttt{COQ\_XML\_LIBRARY\_ROOT} is set, this command creates files containing the logical content in XML format of \textit{qualid}. If the variable is not set, the result is displayed on the standard output.

\textbf{Variants:}

1. Print XML File \textit{string} \textit{qualid}
   This writes the logical content of \textit{qualid} in XML format to files whose prefix is \textit{string}.

Show XML Proof

If the variable \texttt{COQ\_XML\_LIBRARY\_ROOT} is set, this command creates files containing the current proof in progress in XML format. It writes also an XML file made of inner types. If the variable is not set, the result is displayed on the standard output.

\textbf{Variants:}

1. Show XML File \textit{string} \textit{Proof}
   This writes the logical content of \textit{qualid} in XML format to files whose prefix is \textit{string}.

14.5.6 Applications: rendering, searching and publishing

The HELM team at the University of Bologna has developed tools exploiting the XML exportation of CoQ libraries. This covers rendering, searching and publishing tools.

All these tools require a running http server and, if possible, a MathML compliant browser. The procedure to install the suite of tools ultimately allowing rendering and searching can be found on the HELM web site \url{http://helm.cs.unibo.it/library.html}.

It may be easier though to upload your developments on the HELM http server and to re-use the infrastructure running on it. This requires publishing your development. To this aim, follow the instructions on \url{http://mowgli.cs.unibo.it}.

Notice that the HELM server already hosts a copy of the standard library of CoQ and of the CoQ user contributions.

14.5.7 Technical informations

\textbf{CIC with Explicit Named Substitutions}

The exported files are XML encoding of the lambda-terms used by the CoQ system. The implementative details of the CoQ system are hidden as much as possible, so that the XML DTD is a straightforward encoding of the Calculus of (Co)Inductive Constructions.

Nevertheless, there is a feature of the CoQ system that can not be hidden in a completely satisfactory way: discharging (see Sect. 2.4). In CoQ it is possible to open a section, declare variables and use them in the rest of the section as if they were axiom declarations. Once the section is closed, every definition and theorem in the section is discharged by abstracting it over the section variables. Variable declarations as well as section declarations are entirely dropped. Since we are interested in an XML encoding of definitions and theorems as close as possible to those directly provided the user, we do not want to export discharged forms. Exporting non-discharged theorem and definitions together with theorems that rely on the discharged forms obliges the tools that work on the XML encoding to implement discharging to achieve logical consistency. Moreover, the rendering of the files can be misleading, since hyperlinks can
be shown between occurrences of the discharge form of a definition and the non-discharged definition, that are different objects.

To overcome the previous limitations, Claudio Sacerdoti Coen developed in his PhD. thesis an extension of CIC, called Calculus of (Co)Inductive Constructions with Explicit Named Substitutions, that is a slight extension of CIC where discharging is not necessary. The DTD of the exported XML files describes constants, inductive types and variables of the Calculus of (Co)Inductive Constructions with Explicit Named Substitutions. The conversion to the new calculus is performed during the exportation phase.

The following example shows a very small COQ development together with its version in CIC with Explicit Named Substitutions.

# CIC version: #

Section S.
Variable A : Prop.

Definition impl := A -> A.

Theorem t : impl. (* uses the undischarged form of impl *)
  Proof.
    exact (fun (a:A) => a).
  Qed.

End S.

Theorem t' : (impl False). (* uses the discharged form of impl *)
  Proof.
    exact (t False). (* uses the discharged form of t *)
  Qed.

# Corresponding CIC with Explicit Named Substitutions version: #

Section S.
Variable A : Prop.

Definition impl(A) := A -> A. (* theorems and definitions are explicitly abstracted over the variables. The name is sufficient to completely describe the abstraction *)

Theorem t(A) : impl. (* impl where A is not instantiated *)
  Proof.
    exact (fun (a:A) => a).
  Qed.

End S.

Theorem t'(A) : impl(False/A). (* impl where A is instantiated with False Notice that t’ does not depend on A *)
  Proof.
Further details on the typing and reduction rules of the calculus can be found in Claudio Sacerdoti Coen PhD. dissertation, where the consistency of the calculus is also proved.

The CIC with Explicit Named Substitutions XML DTD

A copy of the DTD can be found in the file “cic.dtd” in the contrib/xml source directory of Coq. The following is a very brief overview of the elements described in the DTD.

<ConstantType> is the root element of the files that correspond to constant types.

<ConstantBody> is the root element of the files that correspond to constant bodies. It is used only for closed definitions and theorems (i.e. when no metavariable occurs in the body or type of the constant)

<CurrentProof> is the root element of the file that correspond to the body of a constant that depends on metavariables (e.g. unfinished proofs)

<Variable> is the root element of the files that correspond to variables

<InductiveTypes> is the root element of the files that correspond to blocks of mutually defined inductive definitions

The elements <LAMBDA>, <CAST>, <PROD>, <REL>, <SORT>, <APPLY>, <VAR>, <META>, <IMPLICIT>, <CONST>, <LETIN>, <MUTIND>, <MUTCONSTRUCT>, <MUTCASE>, <FIX> and <COFIX> are used to encode the constructors of CIC. The sort or type attribute of the element, if present, is respectively the sort or the type of the term, that is a sort because of the typing rules of CIC.

The element <instantiate> correspond to the application of an explicit named substitution to its first argument, that is a reference to a definition or declaration in the environment.

All the other elements are just syntactic sugar.

14.6 Embedded Coq phrases inside \LaTeX documents

When writing a documentation about a proof development, one may want to insert Coq phrases inside a \LaTeX document, possibly together with the corresponding answers of the system. We provide a mechanical way to process such Coq phrases embedded in \LaTeX files: the coq-tex filter. This filter extracts Coq phrases embedded in \LaTeX files, evaluates them, and insert the outcome of the evaluation after each phrase.

Starting with a file file.tex containing Coq phrases, the coq-tex filter produces a file named file.v.tex with the Coq outcome.

There are options to produce the Coq parts in smaller font, italic, between horizontal rules, etc. See the man page of coq-tex for more details.

Remark. This Reference Manual and the Tutorial have been completely produced with coq-tex.
14.7 COQ and GNU EMACS

14.7.1 The COQ Emacs mode

COQ comes with a Major mode for GNU EMACS, coq.el. This mode provides syntax highlighting (assuming your GNU EMACS library provides hilit19.el) and also a rudimentary indentation facility in the style of the Caml GNU EMACS mode.

Add the following lines to your .emacs file:

```lisp
(setq auto-mode-alist (cons '("\.v\$" . coq-mode) auto-mode-alist))
(setq coq-mode-hook "Major mode for editing Coq vernacular.")
(autoload 'coq-mode "coq" "Major mode for editing Coq vernacular." t)
```

The COQ major mode is triggered by visiting a file with extension .v, or manually with the command M-x coq-mode. It gives you the correct syntax table for the COQ language, and also a rudimentary indentation facility:

- pressing TAB at the beginning of a line indents the line like the line above;
- extra TABs increase the indentation level (by 2 spaces by default);
- M-TAB decreases the indentation level.

An inferior mode to run COQ under Emacs, by Marco Maggesi, is also included in the distribution, in file coq-inferior.el. Instructions to use it are contained in this file.

14.7.2 Proof General

Proof General is a generic interface for proof assistants based on Emacs (or XEmacs). The main idea is that the COQ commands you are editing are sent to a COQ toplevel running behind Emacs and the answers of the system automatically inserted into other Emacs buffers. Thus you don’t need to copy-paste the COQ material from your files to the COQ toplevel or conversely from the COQ toplevel to some files.

Proof General is developped and distributed independently of the system COQ. It is freely available at proofgeneral.inf.ed.ac.uk.

14.8 Module specification

Given a COQ vernacular file, the gallina filter extracts its specification (inductive types declarations, definitions, type of lemmas and theorems), removing the proofs parts of the file. The COQ file file.v gives birth to the specification file file.g (where the suffix .g stands for GALLINA).

See the man page of gallina for more details and options.

14.9 Man pages

There are man pages for the commands coqdep, gallina and coq-tex. Man pages are installed at installation time (see installation instructions in file INSTALL, step 6).
Chapter 15

COQ Integrated Development Environment

The COQ Integrated Development Environment is a graphical tool, to be used as a user-friendly replacement to coqtop. Its main purpose is to allow the user to navigate forward and backward into a COQ vernacular file, executing corresponding commands or undoing them respectively.

COQIDE is run by typing the command coqide on the command line. Without argument, the main screen is displayed with an “unnamed buffer”, and with a file name as argument, another buffer displaying the contents of that file. Additionally, coqide accepts the same options as coqtop, given in Chapter 13, the ones having obviously no meaning for COQIDE being ignored. Additionally, coqide accepts the option -enable-geoproof to enable the support for GeoProof.

A sample COQIDE main screen, while navigating into a file Fermat.v, is shown on Figure 15.1. At the top is a menu bar, and a tool bar below it. The large window on the left is displaying the various script buffers. The upper right window is the goal window, where goals to prove are displayed. The lower right window is the message window, where various messages resulting from commands are displayed. At the bottom is the status bar.

15.1 Managing files and buffers, basic edition

In the script window, you may open arbitrarily many buffers to edit. The File menu allows you to open files or create some, save them, print or export them into various formats. Among all these buffers, there is always one which is the current running buffer, whose name is displayed on a green background, which is the one where Coq commands are currently executed.

Buffers may be edited as in any text editor, and classical basic editing commands (Copy/Paste, . . . ) are available in the Edit menu. COQIDE offers only basic editing commands, so if you need more complex editing commands, you may launch your favorite text editor on the current buffer, using the Edit/External Editor menu.

1GeoProof is dynamic geometry software which can be used in conjunction with COQIDE to interactively build a Coq statement corresponding to a geometric figure. More information about GeoProof can be found here: http://home.gna.org/geoproof/
15.2 Interactive navigation into COQ scripts

The running buffer is the one where navigation takes place. The toolbar proposes five basic commands for this. The first one, represented by a down arrow icon, is for going forward executing one command. If that command is successful, the part of the script that has been executed is displayed on a green background. If that command fails, the error message is displayed in the message window, and the location of the error is emphasized by a red underline.

On Figure 15.1, the running buffer is Fermat.v, all commands until the Theorem have been already executed, and the user tried to go forward executing Induction n. That command failed because no such tactic exist (tactics are now in lowercase...), and the wrong word is underlined.

Notice that the green part of the running buffer is not editable. If you ever want to modify something you have to go backward using the up arrow tool, or even better, put the cursor where you want to go back and use the goto button. Unlike with coqtop, you should never use Undo to go backward.

Two additional tool buttons exist, one to go directly to the end and one to go back to the beginning. If you try to go to the end, or in general to run several commands using the goto button, the execution will stop whenever an error is found.

If you ever try to execute a command which happens to run during a long time, and would like to abort it before its termination, you may use the interrupt button (the white cross on a red circle).

Finally, notice that these navigation buttons are also available in the menu, where their keyboard shortcuts are given.
15.3 Try tactics automatically

The menu Try Tactics provides some features for automatically trying to solve the current goal using simple tactics. If such a tactic succeeds in solving the goal, then its text is automatically inserted into the script. There is finally a combination of these tactics, called the proof wizard which will try each of them in turn. This wizard is also available as a tool button (the light bulb). The set of tactics tried by the wizard is customizable in the preferences.

These tactics are general ones, in particular they do not refer to particular hypotheses. You may also try specific tactics related to the goal or one of the hypotheses, by clicking with the right mouse button one the goal or the considered hypothesis. This is the “contextual menu on goals” feature, that may be disabled in the preferences if undesirable.

15.4 Vernacular commands, templates

The Templates menu allows to use shortcuts to insert vernacular commands. This is a nice way to proceed if you are not sure of the spelling of the command you want.

Moreover, this menu offers some templates which will automatic insert a complex command like Fixpoint with a convenient shape for its arguments.
15.5 Queries

We call *query* any vernacular command that does not change the current state, such as Check, SearchAbout, etc. Those commands are of course useless during compilation of a file, hence should not be included in scripts. To run such commands without writing them in the script, CoQIDE offers another input window called the *query window*. This window can be displayed on demand, either by using the Window menu, or directly using shortcuts given in the Queries menu. Indeed, with CoQIDE the simplest way to perform a SearchAbout on some identifier is to select it using the mouse, and pressing F2. This will both make appear the query window and run the SearchAbout in it, displaying the result. Shortcuts F3 and F4 are for Check and Print respectively. Figure 15.2 displays the query window after selection of the word “mult” in the script windows, and pressing F4 to print its definition.

15.6 Compilation

The *Compile* menu offers direct commands to:

- compile the current buffer
- run a compilation using *make*
- go to the last compilation error
- create a makefile using *coq_makefile*.

15.7 Customizations

You may customize your environment using menu Edit/Preferences. A new window will be displayed, with several customization sections presented as a notebook.

The first section is for selecting the text font used for scripts, goal and message windows.

The second section is devoted to file management: you may configure automatic saving of files, by periodically saving the contents into files named #f# for each opened file f. You may also activate the revert feature: in case a opened file is modified on the disk by a third party, CoQIDE may read it again for you. Note that in the case you edited that same file, you will be prompt to choose to either discard your changes or not. The File charset encoding choice is described below in Section 15.8.3

The Externals section allows to customize the external commands for compilation, printing, web browsing. In the browser command, you may use %s to denote the URL to open, for example: mozilla -remote "OpenURL(%s)".

The Tactics Wizard section allows to defined the set of tactics that should be tried, in sequence, to solve the current goal.

The last section is for miscellaneous boolean settings, such as the “contextual menu on goals” feature presented in Section 15.3.

Notice that these settings are saved in the file .coqiderc of your home directory.

A gtk2 accelerator keymap is saved under the name .coqide.keys. This file should not be edited manually: to modify a given menu shortcut, go to the corresponding menu item without releasing the mouse button, press the key you want for the new shortcut, and release the mouse button afterwards.

For experts: it is also possible to set up a specific gtk resource file, under the name .coqide-gtk2rc, following the gtk2 resources syntax http://developer.gnome.org/doc/API/2.0/gtk/gtk-Resource-Files.html. Such a default resource file can be found
in the subdirectory lib/coq/ide of the root installation directory of CoQ (alternatively, it can be found in the subdirectory ide of the source archive of CoQ). You may copy this file into your home directory, and edit it using any text editor, CoQIDE itself for example.

15.8 Using unicode symbols

CoQIDE supports unicode character encoding in its text windows, consequently a large set of symbols is available for notations.

15.8.1 Displaying unicode symbols

You just need to define suitable notations as described in Chapter 12. For example, to use the mathematical symbols ∀ and ∃, you may define

\begin{verbatim}
Notation "∀ x : t, P" :=
(forall x:t, P) (at level 200, x ident).
Notation "∃ x : t, P" :=
(exists x:t, P) (at level 200, x ident).
\end{verbatim}

There exists a small set of such notations already defined, in the file utf8.v of CoQ library, so you may enable them just by Require utf8 inside CoQIDE, or equivalently, by starting CoQIDE with coqide -l utf8.

However, there are some issues when using such unicode symbols: you of course need to use a character font which supports them. In the Fonts section of the preferences, the Preview line displays some unicode symbols, so you could figure out if the selected font is OK. Related to this, one thing you may need to do is choose whether Gtk should use antialiased fonts or not, by setting the environment variable GDK_USE_XFT to 1 or 0 respectively.

15.8.2 Defining an input method for non ASCII symbols

To input an Unicode symbol, a general method is to press both the CONTROL and the SHIFT keys, and type the hexadecimal code of the symbol required, for example 2200 for the ∀ symbol. A list of symbol codes is available at http://www.unicode.org.

Of course, this method is painful for symbols you use often. There is always the possibility to copy-paste a symbol already typed in. Another method is to bind some key combinations for frequently used symbols. For example, to bind keys F11 and F12 to ∀ and ∃ respectively, you may add

\begin{verbatim}
bind "F11" "insert-at-cursor" ("∀")
bind "F12" "insert-at-cursor" ("∃")
\end{verbatim}

to your binding "text" section in .coqide-gtk2rc.

15.8.3 Character encoding for saved files

In the Files section of the preferences, the encoding option is related to the way files are saved.

If you have no need to exchange files with non UTF-8 aware applications, it is better to choose the UTF-8 encoding, since it guarantees that your files will be read again without problems. (This is because when CoQIDE reads a file, it tries to automatically detect its character encoding.)
If you choose something else than UTF-8, then missing characters will be written encoded by \x{.....} or \x{..........} where each dot is an hexadecimal digit: the number between braces is the hexadecimal UNICODE index for the missing character.

## 15.9 Building a custom CoQIDE with user ML code

You can do this as described in Section 14.1 for a custom coq text toplevel, simply by adding option -ide to coqmktop, that is something like

```bash
ccqmktop -ide -byte m_1.cmo ... m_n.cmo
```

or

```bash
ccqmktop -ide -opt m_1.cmx ... m_n.cmx
```
Part V

Addendum to the Reference Manual
Presentation of the Addendum

Here you will find several pieces of additional documentation for the COQ Reference Manual. Each of
this chapters is concentrated on a particular topic, that should interest only a fraction of the COQ users:
that’s the reason why they are apart from the Reference Manual.

Extended pattern-matching  This chapter details the use of generalized pattern-matching. It is con-
tributed by Cristina Cornes and Hugo Herbelin.

Implicit coercions  This chapter details the use of the coercion mechanism. It is contributed by
Amokrane Saïbi.

Program extraction  This chapter explains how to extract in practice ML files from $F_\omega$ terms. It is
contributed by Jean-Christophe Filliâtre and Pierre Letouzey.

Program  This chapter explains the use of the Program vernacular which allows the development of
certified programs in COQ. It is contributed by Matthieu Sozeau and replaces the previous
Program tactic by Catherine Parent.

omega  omega, written by Pierre Crégut, solves a whole class of arithmetic problems.

The ring tactic  This is a tactic to do AC rewriting. This chapter explains how to use it and how it
works. The chapter is contributed by Patrick Loiseleur.

The Setoid_replace tactic  This is a tactic to do rewriting on types equipped with specific (only
partially substitutive) equality. The chapter is contributed by Clément Renard.

Calling external provers  This chapter describes several tactics which call external provers.

Contents

Extended pattern-matching  331
  Patterns ................................................................. 331
  About patterns of parametric types .................................. 334
  Matching objects of dependent types ................................ 335
    Understanding dependencies in patterns .......................... 335
    When the elimination predicate must be provided ............... 336
  Using pattern matching to write proofs ............................ 337
  Pattern-matching on inductive objects involving local definitions .... 338
  Pattern-matching and coercions .................................... 339
  When does the expansion strategy fail? ............................ 339
Implicit Coercions 341
General Presentation .................................................. 341
Classes ................................................................. 341
Coercions ............................................................... 342
Identity Coercions ....................................................... 342
Inheritance Graph ....................................................... 343
Declaration of Coercions ............................................... 343
Coercion qualid : class\(_1\) \(\rightarrow\) class\(_2\) ........................................... 343
Identity Coercion ident : class\(_1\) \(\rightarrow\) class\(_2\) ......................... 344
Displaying Available Coercions ........................................ 345
Print Classes ............................................................. 345
Print Coercions ......................................................... 345
Print Graph ............................................................. 345
Print Coercion Paths class\(_1\) class\(_2\) ......................................... 345
Activating the Printing of Coercions ................................... 345
Set Printing Coercions .................................................. 345
Set Printing Coercion qualid .......................................... 345
Classes as Records ...................................................... 346
Coercions and Sections ................................................. 346
Examples ................................................................. 346

Type Classes 351
Class and Instance declarations ....................................... 351
Binding classes .......................................................... 352
Parameterized Instances ............................................... 353
Building hierarchies ..................................................... 354

Omega: a solver of quantifier-free problems in Presburger Arithmetict 357
Description of omega ................................................... 357
Arithmetical goals recognized by omega ................................ 357
Messages from omega ................................................... 358
Technical data .......................................................... 359
Overview of the tactic .................................................. 359
Overview of the OMEGA decision procedure ....................... 359
Bugs ......................................................................... 360

Micromega: tactics for solving arithmetics goals over ordered rings 361

Extraction of programs in Objective Caml and Haskell 365
Generating ML code ..................................................... 365
Extraction options ....................................................... 366
Setting the target language ............................................. 366
Inlining and optimizations .............................................. 366
Realizing axioms ......................................................... 368
Avoiding conflicts with existing filenames ......................... 369
Differences between COQ and ML type systems ................... 370
Some examples .......................................................... 370
A detailed example: Euclidean division .............................. 371
Another detailed example: Heapsort ........................................ 372
The Standard Library ....................................................... 375
Extraction's horror museum ............................................... 376
Users' Contributions ....................................................... 376

PROGRAM ................................................................. 377
Elaborating programs ....................................................... 377

The ring and field tactic families ...................................... 383
What does this tactic do? ................................................... 383
The variables map .......................................................... 384
Is it automatic? ............................................................... 384
Concrete usage in Coq ..................................................... 384
Adding a ring structure ................................................... 386
How does it work? ........................................................... 389
Dealing with fields .......................................................... 390
Adding a new field structure ............................................. 391
Legacy implementation .................................................... 392
History of ring .............................................................. 395
Discussion ........................................................................ 396

User defined equalities and relations .................................. 397
Relations and morphisms .................................................. 398
Adding new relations and morphisms ................................... 399
Rewriting and non reflexive relations ................................... 401
Rewriting and non symmetric relations ................................ 402
Rewriting in ambiguous setoid contexts .............................. 402
First class setoids and morphisms ....................................... 403
Tactics enabled on user provided relations ......................... 404
Printing relations and morphisms ....................................... 404
Deprecated syntax and backward incompatibilities ............. 405
Rewriting under binders .................................................... 405
Sub-relations .................................................................... 406
Constant unfolding ........................................................... 406

Calling external provers .................................................... 407
The gappa tactic ............................................................... 407

Coq Reference Manual, V8.2pl2, June 29, 2010
Chapter 16

Extended pattern-matching

Cristina Cornes and Hugo Herbelin

This section describes the full form of pattern-matching in COQ terms.

16.1 Patterns

The full syntax of `match` is presented in Figures 1.1 and 1.2. Identifiers in patterns are either constructor names or variables. Any identifier that is not the constructor of an inductive or coinductive type is considered to be a variable. A variable name cannot occur more than once in a given pattern. It is recommended to start variable names by a lowercase letter.

If a pattern has the form \((c \vec{x})\) where \(c\) is a constructor symbol and \(\vec{x}\) is a linear vector of (distinct) variables, it is called **simple**: it is the kind of pattern recognized by the basic version of `match`. On the opposite, if it is a variable \(x\) or has the form \((c \vec{p})\) with \(p\) not only made of variables, the pattern is called **nested**.

A variable pattern matches any value, and the identifier is bound to that value. The pattern "\(\_\)" (called "don’t care" or "wildcard" symbol) also matches any value, but does not bind anything. It may occur an arbitrary number of times in a pattern. Alias patterns written \((pattern as identifier)\) are also accepted. This pattern matches the same values as `pattern` does and `identifier` is bound to the matched value. A pattern of the form `pattern | pattern` is called **disjunctive**. A list of patterns separated with commas is also considered as a pattern and is called **multiple pattern**. However multiple patterns can only occur at the root of pattern-matching equations. Disjunctions of **multiple pattern** are allowed though.

Since extended `match` expressions are compiled into the primitive ones, the expressiveness of the theory remains the same. Once the stage of parsing has finished only simple patterns remain. Re-nesting of pattern is performed at printing time. An easy way to see the result of the expansion is to toggle off the nesting performed at printing (use here `Set Printing Matching`), then by printing the term with `Print` if the term is a constant, or using the command `Check`.

The extended `match` still accepts an optional **elimination predicate** given after the keyword `return`. Given a pattern matching expression, if all the right-hand-sides of `=> (rhs in short)` have the same type, then this type can be sometimes synthesized, and so we can omit the `return` part. Otherwise the predicate after `return` has to be provided, like for the basic `match`. 

Coq Reference Manual, V8.2pl2, June 29, 2010
Let us illustrate through examples the different aspects of extended pattern matching. Consider for example the function that computes the maximum of two natural numbers. We can write it in primitive syntax by:

Coq < Fixpoint max (n m:nat) {struct m} : nat :=
Coq < match n with
Coq < | O => m
Coq < | S n' => match m with
Coq < | O => S n'
Coq < | S m' => S (max n' m')
Coq < end.
Coq < end.

max is recursively defined (decreasing on 2nd argument)

**Multiple patterns**  Using multiple patterns in the definition of \( \text{max} \) allows to write:

Coq < Reset max.
Coq < Fixpoint max (n m:nat) {struct m} : nat :=
Coq < match n, m with
Coq < | O, _ => m
Coq < | S n', O => S n'
Coq < | S n', S m' => S (max n' m')
Coq < end.
Coq < end.

max is recursively defined (decreasing on 2nd argument)

which will be compiled into the previous form.

The pattern-matching compilation strategy examines patterns from left to right. A *match* expression is generated *only* when there is at least one constructor in the column of patterns. E.g. the following example does not build a *match* expression.

Coq < Check (fun x:nat => match x return nat with
Coq < | y => y
Coq < end).

fun x : nat => x
    : nat -> nat

**Aliasing subpatterns**  We can also use "*as ident" to associate a name to a sub-pattern:

Coq < Reset max.
Coq < Fixpoint max (n m:nat) {struct n} : nat :=
Coq < match n, m with
Coq < | O, _ => m
Coq < | S n’ as p, O => p
Coq < | S n', S m' => S (max n' m')
Coq < end.

max is recursively defined (decreasing on 1st argument)
Nested patterns  Here is now an example of nested patterns:

Coq < Fixpoint even (n:nat) : bool :=
Coq <  match n with
Coq <  | O => true
Coq <  | S O => false
Coq <  | S (S n') => even n'
Coq < end.
even is recursively defined (decreasing on 1st argument)

This is compiled into:

Coq < Print even.
even =
  fix even (n : nat) : bool :=
  match n with
  | 0 => true
  | 1 => false
  | S (S n') => even n'
  end
  : nat -> bool
Argument scope is [nat_scope]

In the previous examples patterns do not conflict with, but sometimes it is comfortable to write patterns that admit a non trivial superposition. Consider the boolean function lef that given two natural numbers yields true if the first one is less or equal than the second one and false otherwise. We can write it as follows:

Coq < Fixpoint lef (n m:nat) {struct m} : bool :=
Coq <  match n, m with
Coq <  | O, x => true
Coq <  | x, O => false
Coq <  | S n, S m => lef n m
Coq < end.
lef is recursively defined (decreasing on 2nd argument)

Note that the first and the second multiple pattern superpose because the couple of values 0 0 matches both. Thus, what is the result of the function on those values? To eliminate ambiguity we use the textual priority rule: we consider patterns ordered from top to bottom, then a value is matched by the pattern at the $i$th row if and only if it is not matched by some pattern of a previous row. Thus in the example, 0 0 is matched by the first pattern, and so (lef 0 0) yields true.

Another way to write this function is:

Coq < Reset lef.
Coq < Fixpoint lef (n m:nat) {struct m} : bool :=
Coq <  match n, m with
Coq <  | O, x => true
Coq <  | S n, S m => lef n m
Coq <  | _, _ => false
Coq < end.
lef is recursively defined (decreasing on 2nd argument)
Here the last pattern superposes with the first two. Because of the priority rule, the last pattern will be used only for values that do not match neither the first nor the second one.

Terms with useless patterns are not accepted by the system. Here is an example:

```coq
Coq < Check (fun x:nat =>
Coq <       match x with
Coq <       | O => true
Coq <       | S _ => false
Coq <       | x => true
Coq <       end).
Coq < Coq < Toplevel input, characters 246-255:
> | x => true
> ^^^^^^^^^
Error: This clause is redundant.
```

**Disjunctive patterns** Multiple patterns that share the same right-hand-side can be factorized using the notation `mult_pattern | ... | mult_pattern`. For instance, `max` can be rewritten as follows:

```coq
Coq < Fixpoint max (n m:nat) {struct m} : nat :=
Coq <   match n, m with
Coq <   | S n', S m' => S (max n' m')
Coq <   | 0, p | p, 0 => p
Coq < end.
max is recursively defined (decreasing on 2nd argument)
```

Similarly, factorization of (non necessary multiple) patterns that share the same variables is possible by using the notation `pattern | ... | pattern`. Here is an example:

```coq
Coq < Definition filter_2_4 (n:nat) : nat :=
Coq <   match n with
Coq <   | 2 as m | 4 as m => m
Coq <   | _ => 0
Coq < end.
filter_2_4 is defined
```

Here is another example using disjunctive subpatterns.

```coq
Coq < Definition filter_some_square_corners (p:nat*nat) : nat*nat :=
Coq <   match p with
Coq <   | ((2 as m | 4 as m), (3 as n | 5 as n)) => (m,n)
Coq <   | _ => (0,0)
Coq < end.
filter_some_square_corners is defined
```

### 16.2 About patterns of parametric types

When matching objects of a parametric type, constructors in patterns *do not expect* the parameter arguments. Their value is deduced during expansion. Consider for example the type of polymorphic lists:

```coq
Coq Reference Manual, V8.2pl2, June 29, 2010
```
Coq < Inductive List (A:Set) : Set :=
Coq <  | nil : List A
Coq <  | cons : A -> List A -> List A.
List is defined
List_rect is defined
List_ind is defined
List_rec is defined

We can check the function \textit{tail}:

Coq < Check
Coq <  (fun l:List nat =>
Coq <  match l with
Coq <  | nil => nil nat
Coq <  | cons _ l' => l'
Coq <  end).

\textit{fun l : List nat => match l with}
\textit{  | nil => nil nat}
\textit{  | cons _ l' => l'}
\textit{end} : List nat -> List nat

When we use parameters in patterns there is an error message:

Coq < Check
Coq <  (fun l:List nat =>
Coq <  match l with
Coq <  | nil A => nil nat
Coq <  | cons A _ l' => l'
Coq <  end).

\textit{Coq < Coq < Toplevel input, characters 196-201:}
\textit{  | nil A => nil nat}
\textit{  ^^^^^}
Error: The constructor nil expects no arguments.

16.3 Matching objects of dependent types

The previous examples illustrate pattern matching on objects of non-dependent types, but we can also
use the expansion strategy to destructure objects of dependent type. Consider the type \textit{listn} of lists of
a certain length:

Coq < Inductive listn : nat -> Set :=
Coq <  | niln : listn 0
Coq <  | consn : forall n:nat, nat -> listn n -> listn (S n).
listn is defined
listn_rect is defined
listn_ind is defined
listn_rec is defined

16.3.1 Understanding dependencies in patterns

We can define the function \textit{length} over \textit{listn} by:
Coq < Definition length (n:nat) (l:listn n) := n.
length is defined

Just for illustrating pattern matching, we can define it by case analysis:

Coq < Reset length.
Coq < Definition length (n:nat) (l:listn n) :=
Coq < match l with
Coq < | niln => 0
Coq < | consn n _ _ => S n
Coq < end.
length is defined

We can understand the meaning of this definition using the same notions of usual pattern matching.

### 16.3.2 When the elimination predicate must be provided

The examples given so far do not need an explicit elimination predicate because all the rhs have the same type and the strategy succeeds to synthesize it. Unfortunately when dealing with dependent patterns it often happens that we need to write cases where the type of the rhs are different instances of the elimination predicate. The function \( \text{concat} \) for \( \text{listn} \) is an example where the branches have different type and we need to provide the elimination predicate:

Coq < Fixpoint concat (n:nat) (l:listn n) (m:nat) (l':listn m) {struct l} :
Coq < listn (n + m) :=
Coq < match l in listn n return listn (n + m) with
Coq < | niln => l'
Coq < | consn n' a y => consn (n' + m) a (concat n' y m l')
Coq < end.
concat is recursively defined (decreasing on 2nd argument)

The elimination predicate is \( \text{fun} (n:nat) (l:listn n) => \text{listn} (n+m) \). In general if \( m \) has type \((I\ q_1\ ...\ q_r\ t_1\ ...\ t_s)\) where \( q_1\ ...\ q_r \) are parameters, the elimination predicate should be of the form: \( \text{fun} y_1\ ...\ y_s\ x:\(I\ q_1\ ...\ q_r\ y_1\ ...\ y_s\) => Q \).

In the concrete syntax, it should be written:

\[
\text{match}\ m\ \text{as}\ x\ \text{in}\ (I\ \_\ \_\ ...\ y_1\ ...\ y_s)\ \text{return}\ Q\ \text{with}\ \_\ \_\ \_\ \text{end}
\]

The variables which appear in the \text{in} and \text{as} clause are new and bounded in the property \( Q \) in the \text{return} clause. The parameters of the inductive definitions should not be mentioned and are replaced by \( \_\) .

Recall that a list of patterns is also a pattern. So, when we destructured several terms at the same time and the branches have different type we need to provide the elimination predicate for this multiple pattern. It is done using the same scheme, each term may be associated to an \text{as} and \text{in} clause in order to introduce a dependent product.

For example, an equivalent definition for \( \text{concat} \) (even though the matching on the second term is trivial) would have been:

Coq < Reset concat.
Coq < Fixpoint concat (n:nat) (l:listn n) (m:nat) (l':listn m) {struct l} :
16.4 Using pattern matching to write proofs

Coq < \listn (n + m) :=
Coq < match l in \listn n, l' return \listn (n + m) with
Coq < | niln, x => x
Coq < | consn n' a y, x => consn (n' + m) a (concat n' y m x)
Coq < end.

When the arity of the predicate (i.e. number of abstractions) is not correct Coq raises an error message. For example:

Coq < Fixpoint concat
    (n:nat) (l:listn n) (m:nat) (l':listn m) {struct l} : listn (n + m) :=
Coq < match l, l' with
Coq < | niln, x => x
Coq < | consn n' a y, x => consn (n' + m) a (concat n' y m x)
Coq < end.

Coq < Coq < Coq < Toplevel input, characters 342-343:
> | niln, x => x
> ^
Error:
In environment
concat : forall n : nat,
    listn n -> forall m : nat, listn m -> listn (n + m)
    n : nat
l : listn n
m : nat
l' : listn m
The term "l'" has type "listn m" while it is expected to have type
"listn (?61 + ?62)".

16.4 Using pattern matching to write proofs

In all the previous examples the elimination predicate does not depend on the object(s) matched. But it may depend and the typical case is when we write a proof by induction or a function that yields an object of dependent type. An example of proof using match in given in Section 10.1.

For example, we can write the function buildlist that given a natural number \( n \) builds a list of length \( n \) containing zeros as follows:

Coq < Fixpoint buildlist (n:nat) : listn n :=
Coq < match n return listn n with
Coq < | O => niln
Coq < | S n => consn n 0 (buildlist n)
Coq < end.

We can also use multiple patterns. Consider the following definition of the predicate less-equal Le:

Coq < Inductive LE : nat -> nat -> Prop :=
Coq < | LEO : forall n:nat, LE 0 n
Coq < | LES : forall n m:nat, LE n m -> LE (S n) (S m).

LE is defined
LE_ind is defined
We can use multiple patterns to write the proof of the lemma \(\forall (n \ m : \text{nat}), (\text{LE} \ n \ m) \lor (\text{LE} \ m \ n)\):

\[
\text{Coq} < \text{Fixpoint dec } (n \ m : \text{nat}) \ {\text{struct} \ n} : \text{LE} \ n \ m \ \lor \ \text{LE} \ m \ n :=
\]
\[
\text{Coq} < \text{match } n, m \ \text{return} \ \text{LE} \ n \ m \ \lor \ \text{LE} \ m \ n \ \text{with}
\]
\[
\text{Coq} < \ | \ 0, x \Rightarrow \text{or_introl} \ (\text{LE} \ x \ 0) \ (\text{LEO} \ x) \\
\text{Coq} < \ | \ x, 0 \Rightarrow \text{or_intror} \ (\text{LE} \ x \ 0) \ (\text{LEO} \ x) \\
\text{Coq} < \ | \ S \ n \ \text{as} \ n', S \ m \ \text{as} \ m' \Rightarrow
\]
\[
\text{Coq} < \ \ \text{match} \ \text{dec} \ n \ m \ \text{with}
\]
\[
\text{Coq} < \ | \ \text{or_introl} \ h \Rightarrow \text{or_introl} \ (\text{LE} \ m' \ n') \ (\text{LES} \ n \ m \ h) \\
\text{Coq} < \ | \ \text{or_intror} \ h \Rightarrow \text{or_intror} \ (\text{LE} \ n' \ m') \ (\text{LES} \ m \ n \ h) \\
\text{Coq} < \ \text{end}
\]
\[
\text{dec is recursively defined (decreasing on 1st argument)}
\]

In the example of \(\text{dec}\), the first \textit{match} is dependent while the second is not.

The user can also use \textit{match} in combination with the tactic \textit{refine} (see Section 8.2.2) to build incomplete proofs beginning with a \textit{match} construction.

### 16.5 Pattern-matching on inductive objects involving local definitions

If local definitions occur in the type of a constructor, then there are two ways to match on this constructor. Either the local definitions are skipped and matching is done only on the true arguments of the constructors, or the bindings for local definitions can also be caught in the matching.

Example.

\[
\text{Coq} < \text{Inductive list : nat} \rightarrow \text{Set} :=
\]
\[
\text{Coq} < \ | \ \text{nil} : \text{list} \ 0 \\
\text{Coq} < \ | \ \text{cons} : \forall n : \text{nat}, \ \text{let} \ m := (2 \ast \ n) \ \text{in} \ \text{list} \ m \ \rightarrow \ \text{list} \ (S \ (S \ m)).
\]

In the next example, the local definition is not caught.

\[
\text{Coq} < \text{Fixpoint length } n \ (l : \text{list} \ n) \ {\text{struct} \ l} : \text{nat} :=
\]
\[
\text{Coq} < \ \text{match} \ l \ \text{with}
\]
\[
\text{Coq} < \ | \ \text{nil} \Rightarrow \ 0 \\
\text{Coq} < \ | \ \text{cons} \ n \ \text{10} \Rightarrow \ S \ (\text{length} \ (2 \ast \ n) \ \text{10}) \\
\text{Coq} < \ \text{end}
\]
\[
\text{length is recursively defined (decreasing on 2nd argument)}
\]

But in this example, it is.

\[
\text{Coq} < \text{Fixpoint length’ } n \ (l : \text{list} \ n) \ {\text{struct} \ l} : \text{nat} :=
\]
\[
\text{Coq} < \ \text{match} \ l \ \text{with}
\]
\[
\text{Coq} < \ | \ \text{nil} \Rightarrow \ 0 \\
\text{Coq} < \ | \ \text{cons} \ _\ m \ \text{10} \Rightarrow \ S \ (\text{length’} \ m \ \text{10}) \\
\text{Coq} < \ \text{end}
\]
\[
\text{length’ is recursively defined (decreasing on 2nd argument)}
\]

**Remark:** for a given matching clause, either none of the local definitions or all of them can be caught.
16.6 Pattern-matching and coercions

If a mismatch occurs between the expected type of a pattern and its actual type, a coercion made from constructors is sought. If such a coercion can be found, it is automatically inserted around the pattern.

Example:

```coq
Coq < Inductive I : Set :=
Coq <   | C1 : nat -> I
Coq <   | C2 : I -> I.
I is defined
I_rect is defined
I_ind is defined
I_rec is defined
Coq < Coercion C1 : nat ~> I.
C1 is now a coercion
Coq < Check (fun x => match x with
Coq <     | C2 O => 0
Coq <     | _ => 0
Coq < end).
fun x : I =>
match x with
| C1 _ => 0
| C2 (C1 0) => 0
| C2 (C1 (S _)) => 0
| C2 (C2 _) => 0
end :
    I ~> nat
```

16.7 When does the expansion strategy fail?

The strategy works very like in ML languages when treating patterns of non-dependent type. But there are new cases of failure that are due to the presence of dependencies.

The error messages of the current implementation may be sometimes confusing. When the tactic fails because patterns are somehow incorrect then error messages refer to the initial expression. But the strategy may succeed to build an expression whose sub-expressions are well typed when the whole expression is not. In this situation the message makes reference to the expanded expression. We encourage users, when they have patterns with the same outer constructor in different equations, to name the variable patterns in the same positions with the same name. E.g. to write \((\text{cons } n \ O \ x) \Rightarrow e_1\) and \((\text{cons } n \ _ \ x) \Rightarrow e_2\) instead of \((\text{cons } n \ O \ x) \Rightarrow e_1\) and \((\text{cons } n' \ _ \ x') \Rightarrow e_2\). This helps to maintain certain name correspondence between the generated expression and the original.

Here is a summary of the error messages corresponding to each situation:

Error messages:

1. The constructor \textit{ident} expects \textit{num} arguments

   The variable \textit{ident} is bound several times in pattern \textit{term}

   Found a constructor of inductive type \textit{term} while a constructor of \textit{term} is expected
Patterns are incorrect (because constructors are not applied to the correct number of the arguments, because they are not linear or they are wrongly typed).

2. Non exhaustive pattern-matching

The pattern matching is not exhaustive.

3. The elimination predicate term should be of arity num (for non dependent case) or num (for dependent case)

The elimination predicate provided to match has not the expected arity.

4. Unable to infer a match predicate

Either there is a type incompatibility or the problem involves dependencies

There is a type mismatch between the different branches. The user should provide an elimination predicate.
Chapter 17

Implicit Coercions

Amokrane Saïbi

17.1 General Presentation

This section describes the inheritance mechanism of COQ. In COQ with inheritance, we are not interested in adding any expressive power to our theory, but only convenience. Given a term, possibly not typable, we are interested in the problem of determining if it can be well typed modulo insertion of appropriate coercions. We allow to write:

- \( f \ a \) where \( f : \forall x : A, B \) and \( a : A' \) when \( A' \) can be seen in some sense as a subtype of \( A \).
- \( x : A \) when \( A \) is not a type, but can be seen in a certain sense as a type: set, group, category etc.
- \( f \ a \) when \( f \) is not a function, but can be seen in a certain sense as a function: bijection, functor, any structure morphism etc.

17.2 Classes

A class with \( n \) parameters is any defined name with a type \( \forall (x_1 : A_1) \ldots (x_n : A_n), s \) where \( s \) is a sort. Thus a class with parameters is considered as a single class and not as a family of classes. An object of a class \( C \) is any term of type \( C \ t_1 \ldots t_n \). In addition to these user-classes, we have two abstract classes:

- Sortclass, the class of sorts; its objects are the terms whose type is a sort.
- Funclass, the class of functions; its objects are all the terms with a functional type, i.e. of form \( \forall x : A, B \).

Formally, the syntax of a classes is defined on Figure 17.1.
17 Implicit Coercions

A name \( f \) can be declared as a coercion between a source user-class \( C \) with \( n \) parameters and a target class \( D \) if one of these conditions holds:

- \( D \) is a user-class, then the type of \( f \) must have the form \( \forall (x_1 : A_1) \ldots (x_n : A_n)(y : C x_1 \ldots x_n), D u_1 \ldots u_m \) where \( m \) is the number of parameters of \( D \).
- \( D \) is Funclass, then the type of \( f \) must have the form \( \forall (x_1 : A_1) \ldots (x_n : A_n)(y : C x_1 \ldots x_n)(x : A), B \).
- \( D \) is Sortclass, then the type of \( f \) must have the form \( \forall (x_1 : A_1) \ldots (x_n : A_n)(y : C x_1 \ldots x_n), s \) with \( s \) a sort.

We then write \( f : C \rightarrow D \). The restriction on the type of coercions is called the uniform inheritance condition. Remark that the abstract classes Funclass and Sortclass cannot be source classes.

To coerce an object \( t : C t_1 \ldots t_n \) of \( C \) towards \( D \), we have to apply the coercion \( f \) to it; the obtained term \( f t_1 \ldots t_n t \) is then an object of \( D \).

17.4 Identity Coercions

Identity coercions are special cases of coercions used to go around the uniform inheritance condition. Let \( C \) and \( D \) be two classes with respectively \( n \) and \( m \) parameters and \( f : \forall (x_1 : T_1) \ldots (x_k : T_k)(y : C u_1 \ldots u_n), D v_1 \ldots v_m \) a function which does not verify the uniform inheritance condition. To declare \( f \) as coercion, one has first to declare a subclass \( C' \) of \( C \):

\[
C' \equiv \text{fun} (x_1 : T_1) \ldots (x_k : T_k) => C u_1 \ldots u_n
\]

We then define an identity coercion between \( C' \) and \( C \):

\[
\text{Id}_{C'}_C \equiv \text{fun} (x_1 : T_1) \ldots (x_k : T_k)(y : C' x_1 \ldots x_k) => (y : C u_1 \ldots u_n)
\]

We can now declare \( f \) as coercion from \( C' \) to \( D \), since we can “cast” its type as \( \forall (x_1 : T_1) \ldots (x_k : T_k)(y : C' x_1 \ldots x_k), D v_1 \ldots v_m \).

The identity coercions have a special status: to coerce an object \( t : C' t_1 \ldots t_k \) of \( C' \) towards \( C \), we do not have to insert explicitly \( \text{Id}_{C'}_C \) since \( \text{Id}_{C'}_C t_1 \ldots t_k \) is convertible with \( t \). However we “rewrite” the type of \( t \) to become an object of \( C \); in this case, it becomes \( C u_1^* \ldots u_k^* \) where each \( u_i^* \) is the result of the substitution in \( u_i \) of the variables \( x_j \) by \( t_j \).
17.5 Inheritance Graph

Coercions form an inheritance graph with classes as nodes. We call coercion path an ordered list of coercions between two nodes of the graph. A class $C$ is said to be a subclass of $D$ if there is a coercion path in the graph from $C$ to $D$; we also say that $C$ inherits from $D$. Our mechanism supports multiple inheritance since a class may inherit from several classes, contrary to simple inheritance where a class inherits from at most one class. However there must be at most one path between two classes. If this is not the case, only the oldest one is valid and the others are ignored. So the order of declaration of coercions is important.

We extend notations for coercions to coercion paths. For instance $[f_1;..;f_k] : C \rightarrow D$ is the coercion path composed by the coercions $f_1..f_k$. The application of a coercion path to a term consists of the successive application of its coercions.

17.6 Declaration of Coercions

17.6.1 Coercion $\text{qualid} : \text{class}_1 \rightarrow \rightarrow \text{class}_2$.

Declares the construction denoted by $\text{qualid}$ as a coercion between $\text{class}_1$ and $\text{class}_2$.

Error messages:

1. $\text{qualid}$ not declared
2. $\text{qualid}$ is already a coercion
3. Funclass cannot be a source class
4. Sortclass cannot be a source class
5. $\text{qualid}$ is not a function
6. Cannot find the source class of $\text{qualid}$
7. Cannot recognize $\text{class}_1$ as a source class of $\text{qualid}$
8. $\text{qualid}$ does not respect the inheritance uniform condition
9. Found target class $\text{class}$ instead of $\text{class}_2$

When the coercion $\text{qualid}$ is added to the inheritance graph, non valid coercion paths are ignored; they are signaled by a warning.

Warning:

1. Ambiguous paths: $[f_1^1;..;f_{n_1}^1] : C_1 \rightarrow \rightarrow D_1$
   
   $\ldots$
   
   $[f_1^m;..;f_{n_m}^m] : C_m \rightarrow \rightarrow D_m$

Variants:

1. Local Coercion $\text{qualid} : \text{class}_1 \rightarrow \rightarrow \text{class}_2$.

Declares the construction denoted by $\text{qualid}$ as a coercion local to the current section.
2. Coercion \texttt{ident := term} \\
This defines \texttt{ident} just like Definition \texttt{ident := term}, and then declares \texttt{ident} as a coercion between its source and its target.

3. Coercion \texttt{ident := term : type} \\
This defines \texttt{ident} just like Definition \texttt{ident : type := term}, and then declares \texttt{ident} as a coercion between its source and its target.

4. Local Coercion \texttt{ident := term} \\
This defines \texttt{ident} just like \texttt{Let ident := term}, and then declares \texttt{ident} as a coercion between its source and its target.

5. Assumptions can be declared as coercions at declaration time. This extends the grammar of declarations from Figure 1.3 as follows:

\begin{verbatim}
declaration ::= declaration_keyword assums .
assums ::= simple_assums |
         (simple_assums) ... (simple_assums)
simple_assums ::= ident ... ident : [>] term
\end{verbatim}

If the extra \texttt{>} is present before the type of some assumptions, these assumptions are declared as coercions.

6. Constructors of inductive types can be declared as coercions at definition time of the inductive type. This extends and modifies the grammar of inductive types from Figure 1.3 as follows:

\begin{verbatim}
inductive ::= Inductive \texttt{ind_body} with ... with \texttt{ind_body} .
         | CoInductive \texttt{ind_body} with ... with \texttt{ind_body} .
ind_body ::= ident [binderlet ... binderlet] : term :=
            [ | ] constructor | ... | constructor
constructor ::= ident [binderlet ... binderlet] [:>] term
\end{verbatim}

Especially, if the extra \texttt{>} is present in a constructor declaration, this constructor is declared as a coercion.

17.6.2 Identity Coercion \texttt{ident : class} \texttt{1 =>> class} \texttt{2}.

We check that \texttt{class} \texttt{1} is a constant with a value of the form \(fun (x_1 : T_1) .. (x_n : T_n) => (class_2 t_1..t_m)\) where \(m\) is the number of parameters of \texttt{class} \texttt{2}. Then we define an identity function with the type \(forall (x_1 : T_1) .. (x_n : T_n)(y : class_1 x_1..x_n), class_2 t_1..t_m,\) and we declare it as an identity coercion between \texttt{class} \texttt{1} and \texttt{class} \texttt{2}.

Error messages:

1. \texttt{class} \texttt{1} must be a transparent constant

Variants:
1. **Local Identity Coercion** \( \text{ident} : \text{ident}_1 \rightarrow \text{ident}_2 \).
   Idem but locally to the current section.

2. **SubClass** \( \text{ident} := \text{type} \).
   If \( \text{type} \) is a class \( \text{ident}' \) applied to some arguments then \( \text{ident} \) is defined and an identity coercion of name \( \text{Id}\_\text{ident}_\_\text{ident}' \) is declared. Otherwise said, this is an abbreviation for
   ```
   \text{Definition} \ \text{ident} := \text{type}.
   ```
   followed by
   ```
   \text{Identity Coercion} \ \text{Id}\_\text{ident}_\_\text{ident}' : \text{ident} \rightarrow \text{ident}'.
   ```

3. **Local SubClass** \( \text{ident} := \text{type} \).
   Same as before but locally to the current section.

### 17.7 Displaying Available Coercions

#### 17.7.1 Print Classes.
Print the list of declared classes in the current context.

#### 17.7.2 Print Coercions.
Print the list of declared coercions in the current context.

#### 17.7.3 Print Graph.
Print the list of valid coercion paths in the current context.

#### 17.7.4 Print Coercion Paths \( \text{class}_1 \ \text{class}_2 \).
Print the list of valid coercion paths from \( \text{class}_1 \) to \( \text{class}_2 \).

### 17.8 Activating the Printing of Coercions

#### 17.8.1 Set Printing Coercions.
This command forces all the coercions to be printed. Conversely, to skip the printing of coercions, use
```
\text{Unset Printing Coercions}.
```
By default, coercions are not printed.

#### 17.8.2 Set Printing Coercion \( \text{qualid} \).
This command forces coercion denoted by \( \text{qualid} \) to be printed. To skip the printing of coercion \( \text{qualid} \), use
```
\text{Unset Printing Coercion} \ \text{qualid}.
```
By default, a coercion is never printed.
17.9 Classes as Records

We allow the definition of Structures with Inheritance (or classes as records) by extending the existing Record macro (see Section 2.1). Its new syntax is:

```
Record [>] ident binderlet : sort := [ident_0] {  
  ident_1 [::>] term_1 ;  
  ...  
  ident_n [::>] term_n }.
```

The identifier `ident` is the name of the defined record and `sort` is its type. The identifier `ident_0` is the name of its constructor. The identifiers `ident_1`, ..., `ident_n` are the names of its fields and `term_1`, ..., `term_n` their respective types. The alternative `[::>]` is `::` or `::>`. If `ident_i ::> term_i`, then `ident_i` is automatically declared as coercion from `ident` to the class of `term_i`. Remark that `ident_i` always verifies the uniform inheritance condition. If the optional `::>` before `ident` is present, then `ident_0` (or the default name `Build_ident` if `ident_0` is omitted) is automatically declared as a coercion from the class of `term_n` to `ident` (this may fail if the uniform inheritance condition is not satisfied).

Remark: The keyword Structure is a synonym of Record.

17.10 Coercions and Sections

The inheritance mechanism is compatible with the section mechanism. The global classes and coercions defined inside a section are redefined after its closing, using their new value and new type. The classes and coercions which are local to the section are simply forgotten. Coercions with a local source class or a local target class, and coercions which do not verify the uniform inheritance condition any longer are also forgotten.

17.11 Examples

There are three situations:

- `f a` is ill-typed where `f : forall x : A, B` and `a : A'`. If there is a coercion path between `A'` and `A`, `f a` is transformed into `f a'` where `a'` is the result of the application of this coercion path to `a`.

We first give an example of coercion between atomic inductive types

Coq < Definition bool_in_nat (b:bool) := if b then 0 else 1.
bool_in_nat is defined
bool_in_nat is now a coercion
Coq < Check (0 = true).
0 = true  
  : Prop
Coq < Set Printing Coercions.
Coq < Check (0 = true).
0 = bool_in_nat true  
  : Prop
Warning: “Check true=0.” fails. This is “normal” behaviour of coercions. To validate true=0, the coercion is searched from nat to bool. There is none.

We give an example of coercion between classes with parameters.

```
Coq < Parameters
Coq < (C : nat -> Set) (D : nat -> bool -> Set) (E : bool -> Set).
C is assumed
D is assumed
E is assumed
Coq < Parameter f : forall n:nat, C n -> D (S n) true.
f is assumed
Coq < Coercion f : C >-> D.
f is now a coercion
Coq < Parameter g : forall (n:nat) (b:bool), D n b -> E b.
g is assumed
Coq < Coercion g : D >-> E.
g is now a coercion
Coq < Parameter c : C 0.
c is assumed
Coq < Parameter T : E true -> nat.
T is assumed
Coq < Check (T c).
T c  : nat
Coq < Set Printing Coercions.
Coq < Check (T c).
T (g 1 true (f 0 c))  : nat
```

We give now an example using identity coercions.

```
Coq < Definition D' (b:bool) := D 1 b.
D' is defined
Coq < Identity Coercion IdD'D : D' >-> D.
Coq < Print IdD'D.
IdD'D = (fun (b : bool) (x : D' b) => x):forall b : bool, D' b -> D 1 b
       : forall b : bool, D' b -> D 1 b
Argument scopes are [bool_scope _]
Coq < Parameter d' : D' true.
d' is assumed
Coq < Check (T d').
T d'  : nat
Coq < Set Printing Coercions.
```
In the case of functional arguments, we use the monotonic rule of sub-typing. Approximatively, to coerce \( t : \forall x : A, B \) towards \( \forall x : A', B' \), one have to coerce \( A' \) towards \( A \) and \( B \) towards \( B' \). An example is given below:

```coq
Coq < Parameters (A B : Set) (h : A -> B).
A is assumed
B is assumed
h is assumed
Coq < Coercion h : A >-> B.
h is now a coercion
Coq < Parameter U : (A -> E true) -> nat.
U is assumed
Coq < Parameter t : B -> C 0.
t is assumed
Coq < Check (U t).
U (fun x : A => t x)
  : nat
Coq < Set Printing Coercions.
Coq < Check (U t).
U (fun x : A => g 1 true (f 0 (t (h x))))
  : nat

Remark the changes in the result following the modification of the previous example.

Coq < Parameter U' : (C 0 -> B) -> nat.
U' is assumed
Coq < Parameter t' : E true -> A.
t' is assumed
Coq < Check (U' t')..
U' (fun x : C 0 => t' x)
  : nat
Coq < Set Printing Coercions.
Coq < Check (U' t').
U' (fun x : C 0 => h (t' (g 1 true (f 0 x)))))
  : nat
```

- An assumption \( x : A \) when \( A \) is not a type, is ill-typed. It is replaced by \( x : A' \) where \( A' \) is the result of the application to \( A \) of the coercion path between the class of \( A \) and \( \text{Sortclass} \) if it exists. This case occurs in the abstraction \( \text{fun} \ x : A \Rightarrow t \), universal quantification \( \forall x : A, B \), global variables and parameters of (co-)inductive definitions and functions. In \( \forall x : A, B \), such a coercion path may be applied to \( B \) also if necessary.
\[ \text{Coq} < \text{Parameter Graph : Type.} \]  
\[ \text{Graph is assumed} \]

\[ \text{Coq} < \text{Parameter Node : Graph -> Type.} \]  
\[ \text{Node is assumed} \]

\[ \text{Coq} < \text{Coercion Node : Graph >>-> Sortclass.} \]  
\[ \text{Node is now a coercion} \]

\[ \text{Coq} < \text{Parameter G : Graph.} \]  
\[ \text{G is assumed} \]

\[ \text{Coq} < \text{Parameter Arrows : G -> G -> Type.} \]  
\[ \text{Arrows is assumed} \]

\[ \text{Coq} < \text{Check Arrows.} \]  
\[ \text{Arrows} \]  
\[ : G -> G -> \text{Type} \]

\[ \text{Coq} < \text{Parameter fg : G -> G.} \]  
\[ \text{fg is assumed} \]

\[ \text{Coq} < \text{Check fg.} \]  
\[ \text{fg} \]  
\[ : G -> G \]

\[ \text{Coq} < \text{Set Printing Coercions.} \]

\[ \text{Coq} < \text{Check fg.} \]  
\[ \text{fg} \]  
\[ : \text{Node G -> Node G} \]

- \( f a \) is ill-typed because \( f : A \) is not a function. The term \( f \) is replaced by the term obtained by applying to \( f \) the coercion path between \( A \) and \text{Funclass} if it exists.

\[ \text{Coq} < \text{Parameter bij : Set -> Set -> Set.} \]  
\[ \text{bij is assumed} \]

\[ \text{Coq} < \text{Parameter ap : forall A B:Set, bij A B -> A -> B.} \]  
\[ \text{ap is assumed} \]

\[ \text{Coq} < \text{Coercion ap : bij >>-> Funclass.} \]  
\[ \text{ap is now a coercion} \]

\[ \text{Coq} < \text{Parameter b : bij nat nat.} \]  
\[ \text{b is assumed} \]

\[ \text{Coq} < \text{Check (b 0).} \]  
\[ \text{b 0} \]  
\[ : \text{nat} \]

\[ \text{Coq} < \text{Set Printing Coercions.} \]

\[ \text{Coq} < \text{Check (b 0).} \]  
\[ \text{ap nat nat b 0} \]  
\[ : \text{nat} \]

Let us see the resulting graph of this session.
Coq < Print Graph.
[\text{sigT\_of\_sig; sig\_of\_sigT}] : sig -> sig
[\text{sigT\_of\_sig}] : sig -> sigT
[\text{sig\_of\_sigT}] : sigT -> sig
[\text{sig\_of\_sigT; sigT\_of\_sig}] : sigT -> sigT
[\text{bool\_in\_nat}] : bool -> nat
[f] : C -> D
[f; g] : C -> E
[g] : D -> E
[\text{IdD’D}] : D’ -> D
[\text{IdD’D; g}] : D’ -> E
[h] : A -> B
[\text{Node}] : Graph -> Sortclass
[\text{ap}] : bij -> Funclass
Chapter 18

Type Classes

Matthieu Sozeau

The status of Type Classes is (extremely) experimental.

This chapter presents a quick reference of the commands related to type classes. For an actual introduction to type classes, there is a description of the system [126] and the literature on type classes in HASKELL which also applies.

18.1 Class and Instance declarations

The syntax for class and instance declarations is the same as record syntax of COQ:

\[
\text{Class } \text{Id} (\alpha_1 : \tau_1) \cdots (\alpha_n : \tau_n) : \text{sort} := \{ \\
\quad f_1 : \text{type}_1 ; \\
\quad \vdots \\
\quad f_m : \text{type}_m \}. \\
\text{Instance } \text{ident} : \text{Id } \text{term}_1 \cdots \text{term}_n := \{ \\
\quad f_1 := \text{term}_{f_1} ; \\
\quad \vdots \\
\quad f_m := \text{term}_{f_m} \}.
\]

The $\alpha_i : \tau_i$ variables are called the parameters of the class and the $f_k : \text{type}_k$ are called the methods. Each class definition gives rise to a corresponding record declaration and each instance is a regular definition whose name is given by ident and type is an instantiation of the record type.

We’ll use the following example class in the rest of the chapter:

```
Coq < Class EqDec (A : Type) := 
  eqb : A -> A -> bool ;
  eqb_leibniz : forall x y, eqb x y = true -> x = y .
```

Coq Reference Manual, V8.2pl2, June 29, 2010
This class implements a boolean equality test which is compatible with leibniz equality on some type. An example implementation is:

```coq
Coq < Instance unit_EqDec : EqDec unit :=
Coq < { eqb x y := true ;
Coq <  eqb_leibniz x y H :=
Coq <       match x, y return x = y with tt, tt => refl_equal tt end }.
```

If one does not give all the members in the `Instance` declaration, Coq enters the proof-mode and the user is asked to build inhabitants of the remaining fields, e.g.:

```coq
Coq < Instance eq_bool : EqDec bool :=
Coq < { eqb x y := if x then y else negb y }.
Coq < Proof. intros x y H.
1 subgoal

x : bool
y : bool
H : (if x then y else negb y) = true
============================
x = y
Coq <   destruct x ; destruct y ; (discriminate || reflexivity).
Proof completed.
Coq < Defined.
```

One has to take care that the transparency of every field is determined by the transparency of the `Instance` proof. One can use alternatively the `Program Instance` variant which has richer facilities for dealing with obligations.

### 18.2 Binding classes

Once a type class is declared, one can use it in class binders:

```coq
Coq < Definition neqb {A} {eqa : EqDec A} (x y : A) := negb (eqb x y).
neqb is defined
```

When one calls a class method, a constraint is generated that is satisfied only in contexts where the appropriate instances can be found. In the example above, a constraint `EqDec A` is generated and satisfied by `eqa : EqDec A`. In case no satisfying constraint can be found, an error is raised:

```coq
Coq < Definition neqb' (A : Type) (x y : A) := negb (eqb x y).
Toplevel input, characters 47-50:
> Definition neqb' (A : Type) (x y : A) := negb (eqb x y).
> ^^^
Error: Cannot infer the implicit parameter EqDec of eqb.
```
Could not find an instance for "EqDec ?11" in environment:
A : Type
x : A
y : A

The algorithm used to solve constraints is a variant of the eauto tactic that does proof search with a set of lemmas (the instances). It will use local hypotheses as well as declared lemmas in the typeclass_instances database. Hence the example can also be written:

Coq < Definition neqb' A (eqa : EqDec A) (x y : A) := negb (eqb x y).
neqb' is defined

However, the generalizing binders should be used instead as they have particular support for type classes:

• They automatically set the maximally implicit status for type class arguments, making derived functions as easy to use as class methods. In the example above, A and eqa should be set maximally implicit.

• They support implicit quantification on class arguments and partially applied type classes (§18.2.1)

• They support implicit quantification on superclasses (§18.4.1)

18.2.1 Implicit quantification

Implicit quantification is an automatic elaboration of a statement with free variables into a closed statement where these variables are quantified explicitly. Implicit generalization is done only inside binders begining with a backquote ` and the codomain of Instance declarations.

Following the previous example, one can write:

Coq < Definition neqb_impl '{eqa : EqDec A} (x y : A) := negb (eqb x y).
neqb_impl is defined

Here A is implicitly generalized, and the resulting function is equivalent to the one above. One must be careful that all the free variables are generalized, which may result in confusing errors in case of types. In such cases, the context will probably contain some unexpected generalized variable.

The generalizing binders `{ } and `{ } work similarly to their explicit counterparts, only binding the generalized variables implicitly, as maximally-inserted arguments. In these binders, the binding name for the bound object is optional, whereas the type is mandatory, dually to regular binders.

18.3 Parameterized Instances

One can declare parameterized instances as in HASKELL simply by giving the constraints as a binding context before the instance, e.g.:

Coq < Instance prod_eqb `(EA : EqDec A, EB : EqDec B) : EqDec (A * B) :=
Coq < { eqb x y := match x, y with
Coq < | (la, ra), (lb, rb) => andb (eqb la lb) (eqb ra rb)
Coq < end }.
1 subgoal
A : Type
EA : EqDec A
B : Type
EB : EqDec B

\[
\text{forall } x y : A \times B, \\
(\text{let } (la, ra) := x \text{ in let } (lb, rb) := y \text{ in (eqb la lb && eqb ra rb)%bool}) = \\
\text{true} \rightarrow x = y
\]

These instances are used just as well as lemmas in the instance hint database.

### 18.4 Building hierarchies

#### 18.4.1 Superclasses

One can also parameterize classes by other classes, generating a hierarchy of classes and superclasses. In the same way, we give the superclasses as a binding context:

\[
\text{Coq < Class Ord }'(E : \text{EqDec } A) := \\
\text{Coq < } \{ \text{le : } A \rightarrow A \rightarrow \text{bool } \}.
\]

Contrary to HASKELL, we have no special syntax for superclasses, but this declaration is morally equivalent to:

\[
\text{Class }'(E : \text{EqDec } A) \rightarrow \text{Ord } A := \\
\{ \text{le : } A \rightarrow A \rightarrow \text{bool } \}.
\]

This declaration means that any instance of the \text{Ord} class must have an instance of \text{EqDec}. The parameters of the subclass contain at least all the parameters of its superclasses in their order of appearance (here \text{A} is the only one). As we have seen, \text{Ord} is encoded as a record type with two parameters: a type \text{A} and an \text{E} of type \text{EqDec } A. However, one can still use it as if it had a single parameter inside generalizing binders: the generalization of superclasses will be done automatically.

\[
\text{Coq < Definition le_eqb }'(\text{Ord } A) (x y : A) := \text{andb } (\text{le } x y) (\text{le } y x).
\]

In some cases, to be able to specify sharing of structures, one may want to give explicitly the superclasses. It is is possible to do it directly in regular binders, and using the ! modifier in class binders. For example:

\[
\text{Coq < Definition lt }'(\text{eqa : EqDec } A, ! \text{Ord eqa}) (x y : A) := \\
\text{Coq < andb } (\text{le } x y) (\text{neqb } x y).
\]

The ! modifier switches the way a binder is parsed back to the regular interpretation of Coq. In particular, it uses the implicit arguments mechanism if available, as shown in the example.

#### 18.4.2 Substructures

Substructures are components of a class which are instances of a class themselves. They often arise when using classes for logical properties, e.g.:

\[
\text{Coq < Class Reflexive } (A : \text{Type}) (R : \text{relation } A) := \\
\text{Coq < } \text{reflexivity : forall } x, R \times x.
\]

\[
\text{Coq < Class Transitive } (A : \text{Type}) (R : \text{relation } A) := \\
\text{Coq < } \text{transitivity : forall } x y z, R \times y \rightarrow R \times z.
\]
This declares singleton classes for reflexive and transitive relations, (see 1 for an explanation). These may be used as part of other classes:

\[
\begin{array}{l}
\text{Coq} < \text{Class} \text{ PreOrder } (A : \text{Type}) (R : \text{relation } A) := \\
\text{Coq} < \{ \text{PreOrder_Reflexive } :> \text{Reflexive } A \ R ; \\
\text{Coq} < \text{PreOrder_Transitive } :> \text{Transitive } A \ R \}.
\end{array}
\]

The syntax :> indicates that each PreOrder can be seen as a Reflexive relation. So each time a reflexive relation is needed, a preorder can be used instead. This is very similar to the coercion mechanism of Structure declarations. The implementation simply declares each projection as an instance.

One can also declare existing objects or structure projections using the Existing Instance command to achieve the same effect.

### 18.5 Summary of the commands

#### 18.5.1 Class 

Ident binder\_1 ... binder\_n : sort := \{ field\_1 ; ...; field\_k \}.

The Class command is used to declare a type class with parameters binder\_1 to binder\_n and fields field\_1 to field\_k.

**Variants:**

1. Class Ident binder\_1 ... binder\_n : sort := ident\_1 : type\_1. This variant declares a singleton class whose only method is ident\_1. This singleton class is a so-called definitional class, represented simply as a definition ident\_1 ... ident\_n := type\_1 and whose instances are themselves objects of this type. Definitional classes are not wrapped inside records, and the trivial projection of an instance of such a class is convertible to the instance itself. This can be useful to make instances of existing objects easily and to reduce proof size by not inserting useless projections. The class constant itself is declared rigid during resolution so that the class abstraction is maintained.

#### 18.5.2 Instance

Ident binder\_1 ... binder\_n : Class t\_1 ... t\_n [ | priority ] := \{ field\_1 := b\_1 ; ...; field\_i := b\_i \}

The Instance command is used to declare a type class instance named ident of the class Class with parameters t\_1 to t\_n and fields b\_1 to b\_i, where each field must be a declared field of the class. Missing fields must be filled in interactive proof mode.

An arbitrary context of the form binder\_1 ... binder\_n can be put after the name of the instance and before the colon to declare a parameterized instance. An optional priority can be declared, 0 being the highest priority as for auto hints.

**Variants:**

1. Instance Ident binder\_1 ... binder\_n : Class t\_1 ... t\_n [ | priority ] := term This syntax is used for declaration of singleton class instances. It does not include curly braces and one need not even mention the unique field name.

2. Global Instance One can use the Global modifier on instances declared in a section so that their generalization is automatically redeclared after the section is closed.
3. **Program Instance** Switches the type-checking to `PROGRAM` (chapter 22) and uses the obligation mechanism to manage missing fields.

Besides the `Class` and `Instance` vernacular commands, there are a few other commands related to type classes.

### 18.5.3 Existing Instance `ident`

This command adds an arbitrary constant whose type ends with an applied type class to the instance database. It can be used for redeclaring instances at the end of sections, or declaring structure projections as instances. This is almost equivalent to `Hint Resolve ident : typeclass_instances`.

### 18.5.4 Typeclasses Transparent, Opaque `ident_1 . . . ident_n`

This command defines the transparency of `ident_1 . . . ident_n` during type class resolution. It is useful when some constants prevent some unifications and make resolution fail. It is also useful to declare constants which should never be unfolded during proof-search, like fixpoints or anything which does not look like an abbreviation. This can additionally speed up proof search as the typeclass map can be indexed by such rigid constants (see 8.13.1). By default, all constants and local variables are considered transparent. One should take care not to make opaque any constant that is used to abbreviate a type, like `relation A := A -> A -> Prop`.

This is equivalent to `Hint Transparent,Opaque ident : typeclass_instances`.

### 18.5.5 Typeclasses eauto := [debug] [dfs | bfs] [depth]

This command allows to customize the type class resolution tactic, based on a variant of eauto. The flags semantics are:

- **debug** In debug mode, the trace of successfully applied tactics is printed.
- **dfs, bfs** This sets the search strategy to depth-first search (the default) or breadth-first search.
- **depth** This sets the depth of the search (the default is 100).
Chapter 19

Omega: a solver of quantifier-free problems in Presburger Arithmetic

Pierre Crégut

19.1 Description of \texttt{omega}

\texttt{omega} solves a goal in Presburger arithmetic, i.e. a universally quantified formula made of equations and inequations. Equations may be specified either on the type \texttt{nat} of natural numbers or on the type \texttt{Z} of binary-encoded integer numbers. Formulas on \texttt{nat} are automatically injected into \texttt{Z}. The procedure may use any hypothesis of the current proof session to solve the goal.

Multiplication is handled by \texttt{omega} but only goals where at least one of the two multiplicands of products is a constant are solvable. This is the restriction meant by “Presburger arithmetic”.

If the tactic cannot solve the goal, it fails with an error message. In any case, the computation eventually stops.

19.1.1 Arithmetical goals recognized by \texttt{omega}

\texttt{omega} applied only to quantifier-free formulas built from the connectors

\begin{verbatim}
\&\&, /\, \&, \sim, \rightarrow
\end{verbatim}

on atomic formulas. Atomic formulas are built from the predicates

\begin{verbatim}
=, le, lt, gt, ge
\end{verbatim}

on \texttt{nat} or from the predicates

\begin{verbatim}
=, <, <=, >, >=
\end{verbatim}

on \texttt{Z}. In expressions of type \texttt{nat}, \texttt{omega} recognizes

\begin{verbatim}
plus, minus, mult, pred, S, O
\end{verbatim}
and in expressions of type $\mathbb{Z}$, $\omega$ recognizes

$+, -, *, \text{Zsucc}$, and constants.

All expressions of type $\text{nat}$ or $\mathbb{Z}$ not built on these operators are considered abstractly as if they were arbitrary variables of type $\text{nat}$ or $\mathbb{Z}$.

### 19.1.2 Messages from $\omega$

When $\omega$ does not solve the goal, one of the following errors is generated:

**Error messages:**

1. $\omega$ can’t solve this system
   
   This may happen if your goal is not quantifier-free (if it is universally quantified, try intros first; if it contains existentials quantifiers too, $\omega$ is not strong enough to solve your goal).
   
   This may happen also if your goal contains arithmetical operators unknown from $\omega$. Finally, your goal may be really wrong!

2. $\omega$: Not a quantifier-free goal
   
   If your goal is universally quantified, you should first applyintro as many time as needed.

3. $\omega$: Unrecognized predicate or connective: \texttt{ident}

4. $\omega$: Unrecognized atomic proposition: \texttt{prop}

5. $\omega$: Can’t solve a goal with proposition variables

6. $\omega$: Unrecognized proposition

7. $\omega$: Can’t solve a goal with non-linear products

8. $\omega$: Can’t solve a goal with equality on \texttt{type}

### 19.2 Using $\omega$

The $\omega$ tactic does not belong to the core system. It should be loaded by

```
Coq < Require Import Omega.
Coq < Open Scope Z_scope.
```

**Example 3:**

```
Coq < Goal forall m n:Z, 1 + 2 * m <> 2 * n.
  1 subgoal

  ================
  forall m n : Z, 1 + 2 * m <> 2 * n

Coq < intros; omega.
Proof completed.
```
Example 4:

```coq
Coq < Goal forall z:Z, z > 0 -> 2 * z + 1 > z.
1 subgoal

=============
forall z : Z, z > 0 -> 2 * z + 1 > z
```

```
Coq < intro; omega.
Proof completed.
```

19.3 Technical data

19.3.1 Overview of the tactic

- The goal is negated twice and the first negation is introduced as an hypothesis.
- Hypothesis are decomposed in simple equations or inequations. Multiple goals may result from this phase.
- Equations and inequations over `nat` are translated over `Z`, multiple goals may result from the translation of substraction.
- Equations and inequations are normalized.
- Goals are solved by the `OMEGA` decision procedure.
- The script of the solution is replayed.

19.3.2 Overview of the `OMEGA` decision procedure


Here is an overview, look at the original paper for more information.

- Equations and inequations are normalized by division by the GCD of their coefficients.
- Equations are eliminated, using the Banerjee test to get a coefficient equal to one.
- Note that each inequation defines a half space in the space of real value of the variables.
- Inequations are solved by projecting on the hyperspace defined by cancelling one of the variable. They are partitioned according to the sign of the coefficient of the eliminated variable. Pairs of inequations from different classes define a new edge in the projection.
- Redundant inequations are eliminated or merged in new equations that can be eliminated by the Banerjee test.
• The last two steps are iterated until a contradiction is reached (success) or there is no more variable to eliminate (failure).

It may happen that there is a real solution and no integer one. The last steps of the Omega procedure (dark shadow) are not implemented, so the decision procedure is only partial.

19.4 Bugs

• The simplification procedure is very dumb and this results in many redundant cases to explore.
• Much too slow.
• Certainly other bugs! You can report them to

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Chapter 20

Micromega : tactics for solving arithmetics goals over ordered rings

Frédéric Besson and Evgeny Makarov

For using the tactics out-of-the-box, read Section 20.1. Section 20.2 presents some background explaining the proof principle for solving polynomials goals. Section 20.3 explains how to get a complete procedure for linear integer arithmetic.

20.1 The psatz tactic in a hurry

Load the Psatz module (Require Psatz). This module defines the tactics: lia, psatzl D, and psatz D n where D is Z, Q or R and n is an optional integer limiting the proof search depth.

- The psatzl tactic solves linear goals using an embedded (naive) linear programming prover i.e., fourier elimination.

- The psatz tactic solves polynomial goals using John Harrison’s Hol light driver to the external prover cspd. Note that the cspd driver is generating a proof cache thus allowing to rerun scripts even without cspd.

- The lia (linear integer arithmetic) tactic is specialised to solve linear goals over Z. It extends psatzl Z and exploits the discreetness of Z.

These tactics solve propositional formulas parameterised by atomic arithmetics expressions interpreted over a domain $D \in \{Z, Q, R\}$. The syntax of the formulas is the following:

$$
F ::= A \mid P \mid True \mid False \mid F_1 \land F_2 \mid F_1 \lor F_2 \mid F_1 \leftrightarrow F_2 \mid F_1 \rightarrow F_2 \mid \sim F \\
A ::= p_1 = p_2 \mid p_1 > p_2 \mid p_1 < p_2 \mid p_1 \geq p_2 \mid p_1 \leq p_2 \\
p ::= c \mid x \mid \neg p \mid p_1 - p_2 \mid p_1 + p_2 \mid p_1 \times p_2 \mid p^n 
$$

1Sources and binaries can be found at https://projects.coin-or.org/Csdp
where $c$ is a numeric constant, $x \in D$ is a numeric variable and the operators $-, +, \times$, are respectively subtraction, addition, product, $p^n$ is exponentiation by a constant $n$, $P$ is an arbitrary proposition. The following table details for each domain $D \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$ the range of constants $c$ and exponent $n$.

<table>
<thead>
<tr>
<th></th>
<th>$\mathbb{Z}$</th>
<th>$\mathbb{Q}$</th>
<th>$\mathbb{R}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c$</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Q}$</td>
<td>${R1, R0}$</td>
</tr>
<tr>
<td>$n$</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z}$</td>
<td>nat</td>
</tr>
</tbody>
</table>

### 20.2 Positivstellensatz refutations

The name \texttt{psatz} is an abbreviation for \textit{positivstellensatz} – literally positivity theorem – which generalises Hilbert’s \textit{nullstellensatz}. It relies on the notion of \textit{Cone}. Given a (finite) set of polynomials $S$, $\text{Cone}(S)$ is inductively defined as the smallest set of polynomials closed under the following rules:

$$
\begin{align*}
\frac{p \in S}{p \in \text{Cone}(S)} & \\
\frac{p^2 \in \text{Cone}(S)}{p^2 \in \text{Cone}(S)} & \\
\frac{p_1 \in \text{Cone}(S) \quad p_2 \in \text{Cone}(S)}{p_1 \times p_2 \in \text{Cone}(S)} & \\
\frac{\forall \in \{+,*\}}{\text{Cone}(S)} &
\end{align*}
$$

The following theorem provides a proof principle for checking that a set of polynomial inequalities do not have solutions:

**Theorem 1** Let $S$ be a set of polynomials. If $-1$ belongs to $\text{Cone}(S)$ then the conjunction $\bigwedge_{p \in S} p \geq 0$ is unsatisfiable.

A proof based on this theorem is called a \textit{positivstellensatz} refutation. The tactics work as follows. Formulas are normalised into conjunctive normal form $\bigwedge_i C_i$ where $C_i$ has the general form $(\bigwedge_{j \in S} p_j \times 0) \rightarrow \text{False})$ and $\forall \in \{>, \geq, =\}$ for $D \in \{\mathbb{Q}, \mathbb{R}\}$ and $\forall \in \{\geq, =\}$ for $\mathbb{Z}$. For each conjunct $C_i$, the tactic calls an oracle which searches for $-1$ within the cone. Upon success, the oracle returns a \textit{cone expression} that is normalised by the ring tactic (see chapter 23) and checked to be $-1$.

To illustrate the working of the tactic, consider we wish to prove the following Coq goal.

Coq < Goal forall x, -x^2 >= 0 -> x - 1 >= 0 -> False.

Such a goal is solved by intro x; psatz Z. The oracle returns the cone expression $2 \times (x - 1) + x - 1 \times x - 1 + -x^2$ (polynomial hypotheses are printed in bold). By construction, this expression belongs to $\text{Cone}(\{-x^2, x - 1\})$. Moreover, by running ring we obtain $-1$. By Theorem 1, the goal is valid.

The \texttt{psatzl} tactic is searching for \textit{linear} refutations using a fourier elimination\(^3\). As a result, this tactic explore a subset of the \textit{Cone} defined as:

$$
\text{LinCone}(S) = \left\{ \sum_{p \in S} \alpha_p \times p \mid \alpha_p \text{ are positive constants} \right\}
$$

Basically, the deductive power of \texttt{psatzl} is the combined deductive power of \texttt{ring_simplify} and \texttt{fourier}.

\(^2\)Variants deal with equalities and strict inequalities.

\(^3\)More efficient linear programming techniques could equally be employed.

Coq Reference Manual, V8.2pl2, June 29, 2010
The psatz tactic explores the Cone by increasing degrees – hence the depth parameter $n$. In theory, such a proof search is complete – if the goal is provable the search eventually stops. Unfortunately, the external oracle is using numeric (approximate) optimisation techniques that might miss a refutation.

20.3 lia: the linear integer arithmetic tactic

The tactic lia offers an alternative to the omega and romega tactic (see Chapter 19). It solves goals that omega and romega do not solve, such as the following so-called omega nightmare [120].

```coq
Coq < Goal forall x y, Coq < 27 <= 11 * x + 13 * y <= 45 -> Coq < -10 <= 7 * x - 9 * y <= 4 -> False.
```

The estimation of the relative efficiency of lia vs omega and romega is under evaluation.

High level view of lia. Over $\mathbb{R}$, positivstellensatz refutations are a complete proof principle\(^4\). However, this is not the case over $\mathbb{Z}$. Actually, positivstellensatz refutations are not even sufficient to decide linear integer arithmetics. The canonical exemple is $2 * x = 1 \rightarrow$ False which is a theorem of $\mathbb{Z}$ but not a theorem of $\mathbb{R}$. To remedy this weakness, the lia tactic is using recursively a combination of:

- linear positivstellensatz refutations i.e., psatz\(_Z\);
- cutting plane proofs;
- case split.

Cutting plane proofs are a way to take into account the discreetness of $\mathbb{Z}$ by rounding up (rational) constants up-to the closest integer.

**Theorem 2** Let $p$ be an integer and $c$ a rational constant.

\[ p \geq c \Rightarrow p \geq \lceil c \rceil \]

For instance, from $2 * x = 1$ we can deduce

- $x \geq 1/2$ which cut plane is $x \geq \lceil 1/2 \rceil = 1$;
- $x \leq 1/2$ which cut plane is $x \leq \lfloor 1/2 \rfloor = 0$.

By combining these two facts (in normal form) $x - 1 \geq 0$ and $-x \geq 0$, we conclude by exhibiting a positivstellensatz refutation $(-1 \equiv x - 1 + -x \in \text{Cone}(\{x - 1, x\}))$.

Cutting plane proofs and linear positivstellensatz refutations are a complete proof principle for integer linear arithmetic.

Case split allow to enumerate over the possible values of an expression.

**Theorem 3** Let $p$ be an integer and $c_1$ and $c_2$ integer constants.

\[ c_1 \leq p \leq c_2 \Rightarrow \bigvee_{x \in [c_1, c_2]} p = x \]

Our current oracle tries to find an expression $e$ with a small range $[c_1, c_2]$. We generate $c_2 - c_1$ subgoals which contexts are enriched with an equation $e = i$ for $i \in [c_1, c_2]$ and recursively search for a proof.

\(^4\)In practice, the oracle might fail to produce such a refutation.
Chapter 21

Extraction of programs in Objective Caml and Haskell

Jean-Christophe Filliâtre and Pierre Letouzey

The status of extraction is experimental.

We present here the COQ extraction commands, used to build certified and relatively efficient functional programs, extracting them from the proofs of their specifications. The functional languages available as output are currently OBJECTIVE CAML, HASKELL and SCHEME. In the following, “ML” will be used (abusively) to refer to any of the three.

Differences with old versions. The current extraction mechanism is new for version 7.0 of COQ. In particular, the $F_\omega$ toplevel used as an intermediate step between COQ and ML has been withdrawn. It is also not possible any more to import ML objects in this $F_\omega$ toplevel. The current mechanism also differs from the one in previous versions of COQ: there is no more an explicit toplevel for the language (formerly called FML).

21.1 Generating ML code

The next two commands are meant to be used for rapid preview of extraction. They both display extracted term(s) inside COQ.

**Extraction** `qualid`.

Extracts one constant or module in the COQ toplevel.

**Recursive Extraction** `qualid_1 ... qualid_n`.

Recursive extraction of all the globals (or modules) `qualid_1 ... qualid_n` and all their dependencies in the COQ toplevel.

All the following commands produce real ML files. User can choose to produce one monolithic file or one file per COQ library.
Extraction "file" qualid₁ ... qualidₙ.

Recursive extraction of all the globals (or modules) qualid₁ ... qualidₙ, and all their dependencies in one monolithic file file. Global and local identifiers are renamed according to the chosen ML language to fulfill its syntactic conventions, keeping original names as much as possible.

Extraction Library ident.

Extraction of the whole Coq library ident.v to an ML module ident.ml. In case of name clash, identifiers are here renamed using prefixes coq_ or Coq_ to ensure a session-independent renaming.

Recursive Extraction Library ident.

Extraction of the Coq library ident.v and all other modules ident.v depends on.

The list of globals qualidᵢ does not need to be exhaustive: it is automatically completed into a complete and minimal environment.

21.2 Extraction options

21.2.1 Setting the target language

The ability to fix target language is the first and more important of the extraction options. Default is Ocaml. Besides Haskell and Scheme, another language called Toplevel is provided. It is a pseudo-Ocaml, with no renaming on global names: so names are printed as in Coq. This third language is available only at the Coq Toplevel.

Extraction Language Ocaml.

Extraction Language Haskell.

Extraction Language Scheme.

Extraction Language Toplevel.

21.2.2 Inlining and optimizations

Since Objective Caml is a strict language, the extracted code has to be optimized in order to be efficient (for instance, when using induction principles we do not want to compute all the recursive calls but only the needed ones). So the extraction mechanism provides an automatic optimization routine that will be called each time the user want to generate Ocaml programs. Essentially, it performs constants inlining and reductions. Therefore some constants may not appear in resulting monolithic Ocaml program (a warning is printed for each such constant). In the case of modular extraction, even if some inlining is done, the inlined constant are nevertheless printed, to ensure session-independent programs.

Concerning Haskell, such optimizations are less useful because of lazyness. We still make some optimizations, for example in order to produce more readable code.

All these optimizations are controled by the following Coq options:

Set Extraction Optimize.
21.2 Extraction options

Unset Extraction Optimize.

Default is Set. This control all optimizations made on the ML terms (mostly reduction of dummy beta/iota redexes, but also simplications on Cases, etc). Put this option to Unset if you want a ML term as close as possible to the Coq term.

Set Extraction AutoInline.

Unset Extraction AutoInline.

Default is Set, so by default, the extraction mechanism feels free to inline the bodies of some defined constants, according to some heuristics like size of bodies, useness of some arguments, etc. Those heuristics are not always perfect, you may want to disable this feature, do it by Unset.

Extraction Inline qualid₁ ... qualidₙ.

Extraction NoInline qualid₁ ... qualidₙ.

In addition to the automatic inline feature, you can now tell precisely to inline some more constants by the Extraction Inline command. Conversely, you can forbid the automatic inlining of some specific constants by the Extraction NoInline command. Those two commands enable a precise control of what is inlined and what is not.

Print Extraction Inline.

Prints the current state of the table recording the custom inlinings declared by the two previous commands.

Reset Extraction Inline.

Puts the table recording the custom inlinings back to empty.

Inlining and printing of a constant declaration. A user can explicitly ask a constant to be extracted by two means:

- by mentioning it on the extraction command line
- by extracting the whole COQ module of this constant.

In both cases, the declaration of this constant will be present in the produced file. But this same constant may or may not be inlined in the following terms, depending on the automatic/custom inlining mechanism.

For the constants non-explicitly required but needed for dependency reasons, there are two cases:

- If an inlining decision is taken, whether automatically or not, all occurrences of this constant are replaced by its extracted body, and this constant is not declared in the generated file.
- If no inlining decision is taken, the constant is normally declared in the produced file.
21.2.3 Realizing axioms

Extraction will fail if it encounters an informative axiom not realized (see Section 21.2.3). A warning will be issued if it encounters an logical axiom, to remind user that inconsistant logical axioms may lead to incorrect or non-terminating extracted terms.

It is possible to assume some axioms while developing a proof. Since these axioms can be any kind of proposition or object or type, they may perfectly well have some computational content. But a program must be a closed term, and of course the system cannot guess the program which realizes an axiom. Therefore, it is possible to tell the system what ML term corresponds to a given axiom.

\textbf{Extract Constant} \texttt{qualid} \Rightarrow \texttt{string}.

Give an ML extraction for the given constant. The \texttt{string} may be an identifier or a quoted string.

\textbf{Extract Inlined Constant} \texttt{qualid} \Rightarrow \texttt{string}.

Same as the previous one, except that the given ML terms will be inlined everywhere instead of being declared via a let.

\textbf{Note} that the \texttt{Extract Inlined Constant} command is sugar for an \texttt{Extract Constant} followed by a \texttt{Extraction Inline}. Hence a \texttt{Reset Extraction Inline} will have an effect on the realized and inlined \texttt{xaxiom}.

Of course, it is the responsability of the user to ensure that the ML terms given to realize the axioms do have the expected types. In fact, the strings containing realizing code are just copied in the extracted files. The extraction recognize whether the realized axiom should become a ML type constant or a ML object declaration.

\textbf{Example:}

\begin{verbatim}
Coq < Axiom X:Set.
X is assumed
Coq < Axiom x:X.
x is assumed
Coq < Extract Constant X => "int".
Coq < Extract Constant x => "0".
\end{verbatim}

Notice that in the case of type scheme axiom (i.e. whose type is an arity, that is a sequence of product finished by a sort), then some type variables has to be given. The syntax is then:

\textbf{Extract Constant} \texttt{qualid string\textsubscript{1} \ldots string\textsubscript{n}} \Rightarrow \texttt{string}.

The number of type variables is checked by the system.

\textbf{Example:}

\begin{verbatim}
Coq < Axiom Y : Set -> Set -> Set.
Y is assumed
Coq < Extract Constant Y "'a' 'b'" => "'a*'b ".
\end{verbatim}

\textbf{Realizing an axiom} via \texttt{Extract Constant} is only useful in the case of an informative axiom (of sort Type or Set). A logical axioms have no computational content and hence will not appears in extracted terms. But a warning is nonetheless issued if extraction encounters a logical axiom. This
warning reminds user that inconsistent logical axioms may lead to incorrect or non-terminating extracted terms.

If an informative axiom has not been realized before an extraction, a warning is also issued and the definition of the axiom is filled with an exception labelled \texttt{AXIOM\ TO\ BE\ REALIZED}. The user must then search these exceptions inside the extracted file and replace them by real code.

The system also provides a mechanism to specify ML terms for inductive types and constructors. For instance, the user may want to use the ML native boolean type instead of Coq one. The syntax is the following:

\begin{verbatim}
Extract Inductive qualid => string [ string ... string ].
\end{verbatim}

Give an ML extraction for the given inductive type. You must specify extractions for the type itself (first \texttt{string}) and all its constructors (between square brackets). The ML extraction must be an ML recursive datatype.

\textbf{Example:} Typical examples are the following:

\begin{verbatim}
Coq < Extract Inductive unit => "unit" [ "()" ].
Coq < Extract Inductive bool => "bool" [ "true" "false" ].
Coq < Extract Inductive sumbool => "bool" [ "true" "false" ].
\end{verbatim}

If an inductive constructor or type has arity 2 and the corresponding string is enclosed by parenthesis, then the rest of the string is used as infix constructor or type.

\begin{verbatim}
Coq < Extract Inductive list => "list" [ "[]" "(::)" ].
Toplevel input, characters 18-22:
> Extract Inductive list => "list" [ "[]" "(::)" ].
> ^^^^^
Error: The reference list was not found in the current environment.
Coq < Extract Inductive prod => "(*)" [ "(,)"].
\end{verbatim}

\subsection*{21.2.4 Avoiding conflicts with existing filenames}

When using Extraction Library, the names of the extracted files directly depends from the names of the Coq files. It may happen that these filenames are in conflict with already existing files, either in the standard library of the target language or in other code that is meant to be linked with the extracted code. For instance the module \texttt{List} exists both in Coq and in Ocaml. It is possible to instruct the extraction not to use particular filenames.

\begin{verbatim}
Extraction Blacklist ident ... ident .
\end{verbatim}

Instruct the extraction to avoid using these names as filenames for extracted code.

\begin{verbatim}
Print Extraction Blacklist.
\end{verbatim}

Show the current list of filenames the extraction should avoid.

\begin{verbatim}
Reset Extraction Blacklist.
\end{verbatim}

Allow the extraction to use any filename.

\begin{verbatim}
For Ocaml, a typical use of these commands is Extraction Blacklist String List.
\end{verbatim}
21.3 Differences between COQ and ML type systems

Due to differences between COQ and ML type systems, some extracted programs are not directly typable in ML. We now solve this problem (at least in Ocaml) by adding when needed some unsafe casting \texttt{Obj.magic}, which give a generic type \texttt{a} to any term.

For example, Here are two kinds of problem that can occur:

- If some part of the program is \texttt{very} polymorphic, there may be no ML type for it. In that case the extraction to ML works all right but the generated code may be refused by the ML type-checker. A very well known example is the \texttt{distr-pair} function:

\begin{verbatim}
Definition dp :=
  fun (A B:Set)(x:A)(y:B)(f:forall C:Set, C->C) => (f A x, f B y).
\end{verbatim}

In Ocaml, for instance, the direct extracted term would be:

\begin{verbatim}
let dp x y f = Pair((f () x),(f () y))
\end{verbatim}

and would have type:

\begin{verbatim}
dp : \texttt{\texttt{a} -> \texttt{a} -> (\texttt{unit} -> \texttt{a} -> \texttt{b} -> (\texttt{b,\texttt{b}) prod}}
\end{verbatim}

which is not its original type, but a restriction.

We now produce the following correct version:

\begin{verbatim}
let dp x y f = Pair ((\texttt{Obj.magic f () x}), (\texttt{Obj.magic f () y}))
\end{verbatim}

- Some definitions of COQ may have no counterpart in ML. This happens when there is a quantification over types inside the type of a constructor; for example:

\begin{verbatim}
\end{verbatim}

which corresponds to the definition of an ML dynamic type. In Ocaml, we must cast any argument of the constructor \texttt{dummy}.

Even with those unsafe castings, you should never get error like “segmentation fault”. In fact even if your program may seem ill-typed to the Ocaml type-checker, it can’t go wrong: it comes from a Coq well-typed terms, so for example inductives will always have the correct number of arguments, etc.

More details about the correctness of the extracted programs can be found in [91].

We have to say, though, that in most “realistic” programs, these problems do not occur. For example all the programs of Coq library are accepted by Caml type-checker without any \texttt{Obj.magic} (see examples below).

21.4 Some examples

We present here two examples of extractions, taken from the COQ Standard Library. We choose OBJECTIVE CAML as target language, but all can be done in the other dialects with slight modifications. We then indicate where to find other examples and tests of Extraction.
21.4 Some examples

21.4.1 A detailed example: Euclidean division

The file *Euclid* contains the proof of Euclidean division (theorem `eucl_dev`). The natural numbers defined in the example files are unary integers defined by two constructors \( \text{O} \) and \( \text{S} \): 

\[
\text{Coq} \ < \ \text{Inductive} \ \text{nat} : \ \text{Set} := \\
\text{Coq} \ < \ \text{| O : nat} \\
\text{Coq} \ < \ \text{| S : nat -> nat}.
\]

This module contains a theorem `eucl_dev`, and its extracted term is of type 

\[
\text{forall } b:\text{nat}, \ b > 0 \rightarrow \text{forall } a:\text{nat}, \ \text{diveucl } a \ b
\]

where `diveucl` is a type for the pair of the quotient and the modulo. We can now extract this program to **OBJECTIVE CAML**:

\[
\text{Coq} \ < \ \text{Require Import Euclid.} \\
\text{Coq} \ < \ \text{Extraction Inline Wf_nat.gt_wf_rec Wf_nat.lt_wf_rec.} \\
\text{Coq} \ < \ \text{Recursive Extraction eucl_dev.} \\
\text{Coq} \ < \ \text{type __ = Obj.t} \\
\text{let __ = let rec f _ = Obj.repr f in Obj.repr f} \\
\text{type nat =} \\
\text{| O} \\
\text{| S of nat} \\
\text{type sumbool =} \\
\text{| Left} \\
\text{| Right} \\
\text{(** val minus : nat -> nat -> nat **)} \\
\text{let rec minus n m =} \\
\text{match n with} \\
\text{ | O -> n} \\
\text{ | S k -> (match m with} \\
\text{ | O -> n} \\
\text{ | S l -> minus k l)} \\
\text{(** val le_lt_dec : nat -> nat -> sumbool **)} \\
\text{let rec le_lt_dec n m =} \\
\text{match n with} \\
\text{ | O -> Left} \\
\text{ | S n0 -> (match m with} \\
\text{ | O -> Right} \\
\text{ | S m0 -> le_lt_dec n0 m0)} \\
\text{(** val le_gt_dec : nat -> nat -> sumbool **)} \\
\text{let le_gt_dec n m =} \\
\text{le_lt_dec n m} \\
\text{(** val induction_ltof2 :} \\
\text{ ('a1 -> nat) -> ('a1 -> ('a1 -> __ -> 'a2) -> 'a2) -> 'a1 -> 'a2 **)} \\
\text{let rec induction_ltof2 f x a =} \\
\text{ x a (fun y _ -> induction_ltof2 f x y)} \\
\text{type diveucl =} \\
\text{ | Divex of nat * nat} \\
\text{(** val eucl_dev : nat -> nat -> diveucl **)} \\
\text{let eucl_dev b a =} \\
\text{ induction_ltof2 (fun m -> m) (fun n h0 ->}
\]
match le_gt_dec b n with
| Left -> let Divex (x, x0) = h0 (minus n b) __ in Divex ((S x), x0)
| Right -> Divex (O, n)) a

The inlining of gt_wf_rec and lt_wf_rec is not mandatory. It only enhances readability of extracted code. You can then copy-paste the output to a file euclid.ml or let Coq do it for you with the following command:

Coq < Extraction "euclid" eucl_dev.
The file euclid.ml has been created by extraction.
The file euclid.mli has been created by extraction.

Let us play the resulting program:

# #use "euclid.ml";;
type sumbool = Left | Right
type nat = O | S of nat
type diveucl = Divex of nat * nat
val minus : nat -> nat -> nat = <fun>
val le_lt_dec : nat -> nat -> sumbool = <fun>
val le_gt_dec : nat -> nat -> sumbool = <fun>
val eucl_dev : nat -> nat -> diveucl = <fun>
# eucl_dev (S (S O)) (S (S (S (S (S O)))));;
- : diveucl = Divex (S (S O), S O)

It is easier to test on OBJECTIVE CAML integers:

# let rec i2n = function 0 -> O | n -> S (i2n (n-1));;
val i2n : int -> nat = <fun>
# let rec n2i = function O -> 0 | S p -> 1+(n2i p);;
val n2i : nat -> int = <fun>
# let div a b =
  let Divex (q,r) = eucl_dev (i2n b) (i2n a) in (n2i q, n2i r);;
div : int -> int -> int * int = <fun>
# div 173 15;;
- : int * int = 11, 8

21.4.2 Another detailed example: Heapsort

The file Heap.v contains the proof of an efficient list sorting algorithm described by Björner. Is is an adaptation of the well-known heapsort algorithm to functional languages. The main function is treesort, whose type is shown below:

Coq < Require Import Heap.

Coq < Check treesort.
treesort : forall (A : Type) (leA eqA : relation A),
  (forall x y : A, (leA x y) + (leA y x)) ->
  forall eqA_dec : forall x y : A, (eqA x y) + (~ eqA x y),
  (forall x y z : A, leA x y -> leA y z -> leA x z) ->
  forall l : list A,
  {m : list A | sort leA m & permutation eqA eqA_dec l m}
Let’s now extract this function:

Coq < Extraction Inline sort_rec is_heap_rec.
Coq < Extraction NoInline list_to_heap.
Coq < Extraction "heapsort" treesort.
The file heapsort.ml has been created by extraction.
The file heapsort.mli has been created by extraction.

One more time, the Extraction Inline and NoInline directives are cosmetic. Without it, everything goes right, but the output is less readable. Here is the produced file heapsort.ml:

type nat =
    | 0
    | S of nat

type 'a sig2 =
    'a
    (* singleton inductive, whose constructor was exist2 *)

type sumbool =
    | Left
    | Right

type 'a list =
    | Nil
    | Cons of 'a * 'a list

type 'a multiset =
    'a -> nat
    (* singleton inductive, whose constructor was Bag *)

type 'a merge_lem =
    'a list
    (* singleton inductive, whose constructor was merge_exist *)

(** val merge : ('a1 -> 'a1 -> sumbool) -> ('a1 -> 'a1 -> sumbool) ->
    'a1 list -> 'a1 list -> 'a1 merge_lem **)  

let rec merge leA_dec eqA_dec l1 l2 =
    match l1 with
    | Nil -> l2
    | Cons (a, l) ->
      let rec f = function
        | Nil -> Cons (a, l)
        | Cons (a0, l3) ->
          (match leA_dec a a0 with
            | Left -> Cons (a,
(merge leA_dec eqA_dec l (Cons (a0, 13)))
| Right -> Cons (a0, (f 13)))

in f l2

type 'a tree =
| Tree_Leaf
| Tree_Node of 'a * 'a tree * 'a tree

type 'a insert_spec =
'a tree
(* singleton inductive, whose constructor was insert_exist *)

type 'a build_heap =
'a tree
(* singleton inductive, whose constructor was heap_exist *)

let rec insert leA_dec eqA_dec t a =
match t with
| Tree_Leaf -> Tree_Node (a, Tree_Leaf, Tree_Leaf)
| Tree_Node (a0, t0, t1) ->
    let h3 = fun x -> insert leA_dec eqA_dec t0 x in
    (match leA_dec a0 a with
    | Left -> Tree_Node (a0, t1, (h3 a))
    | Right -> Tree_Node (a, t1, (h3 a0)))

let rec list_to_heap leA_dec eqA_dec = function
| Nil -> Tree_Leaf
| Cons (a, l0) ->
    insert leA_dec eqA_dec (list_to_heap leA_dec eqA_dec l0) a

let rec heap_to_list leA_dec eqA_dec = function
| Tree_Leaf -> Nil
| Tree_Node (a, t0, t1) -> Cons (a,
    (merge leA_dec eqA_dec (heap_to_list leA_dec eqA_dec t0)
    Coq Reference Manual, V8.2pl2, June 29, 2010
21.4 Some examples

(let treesort leA_dec eqA_dec l =
  heap_to_list leA_dec eqA_dec (list_to_heap leA_dec eqA_dec l))

let treesort leA_dec eqA_dec l =
  heap_to_list leA_dec eqA_dec (list_to_heap leA_dec eqA_dec l)

Let’s test it:

# #use "heapsort.ml";;
val leAdec : 'a -> 'a -> sumbool = <fun>
# let leAdec x y = if x <= y then Left else Right;;
val leAdec : 'a -> 'a -> sumbool = <fun>

Some tests on longer lists (10000 elements) show that the program is quite efficient for Caml code.

21.4.3 The Standard Library

As a test, we propose an automatic extraction of the Standard Library of COQ. In particular, we will find back the two previous examples, Euclid and Heapsort. Go to directory contrib/extraction/test of the sources of COQ, and run commands:
make tree; make

This will extract all Standard Library files and compile them. It is done via many Extraction Module, with some customization (see subdirectory custom).

This test works also with Haskell. In the same directory, run:

make tree; make -f Makefile.haskell

The haskell compiler currently used is hbc. Any other should also work, just adapt the Makefile.haskell. In particular ghc is known to work.

### 21.4.4 Extraction’s horror museum

Some pathological examples of extraction are grouped in the file contrib/extraction/test_extraction.v of the sources of CoQ.

### 21.4.5 Users’ Contributions

Several of the CoQ Users’ Contributions use extraction to produce certified programs. In particular the following ones have an automatic extraction test (just run make in those directories):

- Bordeaux/Additions
- Bordeaux/EXCEPTIONS
- Bordeaux/SearchTrees
- Dyade/BDDS
- Lannion
- Lyon/CIRCUITS
- Lyon/FIRING-SQUAD
- Marseille/CIRCUITS
- Muenchen/Higman
- Nancy/FOUnify
- Rocq/ARITH/Chinese
- Rocq/COC
- Rocq/GRAPHHS
- Rocq/HIGMAN
- Sophia-Antipolis/Stalmarck
- Suresnes/BDD

Lannion, Rocq/HIGMAN and Lyon/CIRCUITS are a bit particular. They are the only examples of developments where Obj.magic are needed. This is probably due to an heavy use of impredicativity. After compilation those two examples run nonetheless, thanks to the correction of the extraction [91].
Chapter 22

PROGRAM

Matthieu Sozeau

The status of PROGRAM is experimental.

We present here the new PROGRAM tactic commands, used to build certified COQ programs, elaborating them from their algorithmic skeleton and a rich specification [125]. It can be sought of as a dual of extraction (see Chapter 21). The goal of PROGRAM is to program as in a regular functional programming language whilst using as rich a specification as desired and proving that the code meets the specification using the whole COQ proof apparatus. This is done using a technique originating from the “Predicate subtyping” mechanism of PVS[122], which generates type-checking conditions while typing a term constrained to a particular type. Here we insert existential variables in the term, which must be filled with proofs to get a complete COQ term. PROGRAM replaces the PROGRAM tactic by Catherine Parent [111] which had a similar goal but is no longer maintained.

The languages available as input are currently restricted to COQ’s term language, but may be extended to OBJECTIVE CAML, HASKELL and others in the future. We use the same syntax as COQ and permit to use implicit arguments and the existing coercion mechanism. Input terms and types are typed in an extended system (RUSSELL) and interpreted into COQ terms. The interpretation process may produce some proof obligations which need to be resolved to create the final term.

22.1 Elaborating programs

The main difference from COQ is that an object in a type $T : \text{Set}$ can be considered as an object of type $\{x : T \mid P\}$ for any wellformed $P : \text{Prop}$. If we go from $T$ to the subset of $T$ verifying property $P$, we must prove that the object under consideration verifies it. RUSSELL will generate an obligation for every such coercion. In the other direction, RUSSELL will automatically insert a projection.

Another distinction is the treatment of pattern-matching. Apart from the following differences, it is equivalent to the standard match operation (see Section 4.5.4).

- Generation of equalities. A match expression is always generalized by the corresponding equality. As an example, the expression:
Coq < match x with
   | 0 => t
   | S n => u
end.

will be first rewrote to:

Coq < (match x as y return (x = y -> _) with
   | 0 => fun H : x = 0 -> t
   | S n => fun H : x = S n -> u
end) (refl_equal n).

This permits to get the proper equalities in the context of proof obligations inside clauses, without
which reasoning is very limited.

- Generation of inequalities. If a pattern intersects with a previous one, an inequality is added in the
  context of the second branch. See for example the definition of `div2` below, where the second
  branch is typed in a context where \(\forall p, \_ \neq S(Sp)\).

- Coercion. If the object being matched is coercible to an inductive type, the corresponding coercion
  will be automatically inserted. This also works with the previous mechanism.

If you do specify a `return` or `in` clause the typechecker will fall back directly to COQ’s usual
typing of dependent pattern-matching.

The next two commands are similar to their standard counterparts Definition (see Section 1.3.2) and
Fixpoint (see Section 1.3.4) in that they define constants. However, they may require the user to prove
some goals to construct the final definitions.

22.1.1 Program Definition `ident := term`.

This command types the value `term` in RUSSELL and generate proof obligations. Once solved using the
commands shown below, it binds the final COQ term to the name `ident` in the environment.

Error messages:
1. `ident` already exists

Variants:
   It interprets the type `term1`, potentially generating proof obligations to be resolved. Once done
   with them, we have a COQ type `term1'`. It then checks that the type of the interpretation of `term2` is
   coercible to `term1'`, and registers `ident` as being of type `term1'` once the set of obligations generated
during the interpretation of `term2` and the aforementioned coercion derivation are solved.

2. Program Definition `ident binder1...binder_n : term1 := term2`.
   This is equivalent to
   Program Definition `ident : forall binder1...binder_n, term1 := fun binder1...binder_n => term2`.

Error messages:
1. In environment ... the term: `term2` does not have type `term1`.
   Actually, it has type `term3`.

See also: Sections 6.9.1, 6.9.2, 8.5.5
22.1.2 Program Fixpoint  

The structural fixpoint operator behaves just like the one of Coq (see Section 1.3.4), except it may also generate obligations. It works with mutually recursive definitions too.

Here we have one obligation for each branch (branches for 0 and (S 0) are automatically generated by the pattern-matching compilation algorithm).

One can use a well-founded order or a measure as termination orders using the syntax:

The measure keyword expects a measure function into the naturals, whereas \texttt{wf} expects a relation.

Caution When defining structurally recursive functions, the generated obligations should have the prototype of the currently defined functional in their context. In this case, the obligations should be transparent (e.g. using Defined) so that the guardedness condition on recursive calls can be checked by the kernel’s type-checker. There is an optimization in the generation of obligations which gets rid of the hypothesis corresponding to the functional when it is not necessary, so that the obligation can be declared opaque (e.g. using Qed). However, as soon as it appears in the context, the proof of the obligation is \textit{required} to be declared transparent.

No such problems arise when using measures or well-founded recursion.

An important point about well-founded and measure-based functions is the following: The recursive prototype of a function of type \texttt{binder_1...binder_n \{ measure m binder_i \}:type_1}, inside the body is \texttt{\{ binder_i' | m x_i' < m x_i \}...binder_n, type_1}. So any arguments appearing before the recursive one are ignored for the recursive calls, hence they are constant.
22.1.3 Program Lemma $ident : type$.

The RUSSELL language can also be used to type statements of logical properties. It will generate obligations, try to solve them automatically and fail if some unsolved obligations remain. In this case, one can first define the lemma’s statement using Program Definition and use it as the goal afterwards. Otherwise the proof will be started with the elobarted version as a goal. The Program prefix can similarly be used as a prefix for Variable, Hypothesis, Axiom etc...

22.2 Solving obligations

The following commands are available to manipulate obligations. The optional identifier is used when multiple functions have unsolved obligations (e.g. when defining mutually recursive blocks). The optional tactic is replaced by the default one if not specified.

- **Obligation Tactic</code> := expr** Sets the default obligation solving tactic applied to all obligations automatically, whether to solve them or when starting to prove one, e.g. using Next.
- **Obligations [of ident]** Displays all remaining obligations.
- **Obligation num [of ident]** Start the proof of obligation num.
- **Next Obligation [of ident]** Start the proof of the next unsolved obligation.
- **Solve Obligations [of ident] [using expr]** Tries to solve each obligation of ident using the given tactic or the default one.
- **Solve All Obligations [using expr]** Tries to solve each obligation of every program using the given tactic or the default one (useful for mutually recursive definitions).
- **Admit Obligations [of ident]** Admits all obligations (does not work with structurally recursive programs).
- **Preterm [of ident]** Shows the term that will be fed to the kernel once the obligations are solved. Useful for debugging.
- **Set Transparent Obligations** Control whether all obligations should be declared as transparent (the default), or if the system should infer which obligations can be declared opaque.

The module Coq.Program.Tactics defines the default tactic for solving obligations called program_simpl. Importing Coq.Program.Program also adds some useful notations, as documented in the file itself.

22.3 Frequently Asked Questions

- **Ill-formed recursive definitions** This error can happen when one tries to define a function by structural recursion on a subset object, which means the Coq function looks like:

  Program Fixpoint $f (x : A | P) := \text{match } x \text{ with } A \ b \Rightarrow f \ b \text{ end.}$

  Supposing $b : A$, the argument at the recursive call to $f$ is not a direct subterm of $x$ as $b$ is wrapped inside an exist constructor to build an object of type $\{x : A | P\}$. Hence the definition is rejected by the guardedness condition checker. However you can do wellfounded recursion on subset objects like this:
Program Fixpoint f (x : A | P) { measure size } :=
  match x with A b => f b end.

You will then just have to prove that the measure decreases at each recursive call. There are three drawbacks though:

1. You have to define the measure yourself;
2. The reduction is a little more involved, although it works using lazy evaluation;
3. Mutual recursion on the underlying inductive type isn’t possible anymore, but nested mutual recursion is always possible.
Chapter 23

The ring and field tactic families

Bruno Barras, Benjamin Grégoire, Assia Mahboubi, Laurent Théry

This chapter presents the tactics dedicated to deal with ring and field equations.

23.1 What does this tactic do?

ring does associative-commutative rewriting in ring and semi-ring structures. Assume you have two binary functions $\oplus$ and $\otimes$ that are associative and commutative, with $\oplus$ distributive on $\otimes$, and two constants 0 and 1 that are unities for $\oplus$ and $\otimes$. A polynomial is an expression built on variables $V_0, V_1, \ldots$ and constants by application of $\oplus$ and $\otimes$.

Let an ordered product be a product of variables $V_{i_1} \otimes \cdots \otimes V_{i_n}$ verifying $i_1 \leq i_2 \leq \cdots \leq i_n$. Let a monomial be the product of a constant and an ordered product. We can order the monomials by the lexicographic order on products of variables. Let a canonical sum be an ordered sum of monomials that are all different, i.e. each monomial in the sum is strictly less than the following monomial according to the lexicographic order. It is an easy theorem to show that every polynomial is equivalent (modulo the ring properties) to exactly one canonical sum. This canonical sum is called the normal form of the polynomial. In fact, the actual representation shares monomials with same prefixes. So what does ring? It normalizes polynomials over any ring or semi-ring structure. The basic use of ring is to simplify ring expressions, so that the user does not have to deal manually with the theorems of associativity and commutativity.

Examples:

1. In the ring of integers, the normal form of $x(3 + yx + 25(1 - z)) + zx$ is $28x + (-24)xz + xxy$.

2. For the classical propositional calculus (or the boolean rings) the normal form is what logicians call disjunctive normal form: every formula is equivalent to a disjunction of conjunctions of atoms. (Here $\oplus$ is $\lor$, $\otimes$ is $\land$, variables are atoms and the only constants are T and F)

ring is also able to compute a normal form modulo monomial equalities. For example, under the hypothesis that $2x^2 = yz + 1$, the normal form of $2(x + 1)x - x - y$ is $x + 1$.

\footnote{based on previous work from Patrick Loiseleur and Samuel Boutin}
23.2 The variables map

It is frequent to have an expression built with + and ×, but rarely on variables only. Let us associate a number to each subterm of a ring expression in the GALLINA language. For example in the ring nat, consider the expression:

\[
\text{(plus (mult (plus (f (5)) x) x) (mult (if b then (4) else (f (3))) (2)))}
\]

As a ring expression, it has 3 subterms. Give each subterm a number in an arbitrary order:

0 \mapsto \text{if b then (4) else (f (3))}
1 \mapsto (f (5))
2 \mapsto x

Then normalize the “abstract” polynomial

\[
((V_1 \otimes V_2) \oplus V_2) \oplus (V_0 \otimes 2)
\]

In our example the normal form is:

\[
(2 \otimes V_0) \oplus (V_1 \otimes V_2) \oplus (V_2 \otimes V_2)
\]

Then substitute the variables by their values in the variables map to get the concrete normal polynomial:

\[
\text{(plus (mult (2) (if b then (4) else (f (3)))) (plus (mult (f (5)) x) (mult x x)))}
\]

23.3 Is it automatic?

Yes, building the variables map and doing the substitution after normalizing is automatically done by the tactic. So you can just forget this paragraph and use the tactic according to your intuition.

23.4 Concrete usage in COQ

The ring tactic solves equations upon polynomial expressions of a ring (or semi-ring) structure. It proceeds by normalizing both hand sides of the equation (w.r.t. associativity, commutativity and distributivity, constant propagation, rewriting of monomials) and comparing syntactically the results.

ring_simplify applies the normalization procedure described above to the terms given. The tactic then replaces all occurrences of the terms given in the conclusion of the goal by their normal forms. If no term is given, then the conclusion should be an equation and both hand sides are normalized. The tactic can also be applied in a hypothesis.

The tactic must be loaded by Require Import Ring. The ring structures must be declared with the Add Ring command (see below). The ring of booleans is predefined; if one wants to use the tactic on nat one must first require the module ArithRing (exported by Arith); for Z, do Require Import ZArithRing or simply Require Import ZArith; for N, do Require Import NArithRing or Require Import NArith.

Example:
Coq < Require Import ZArith.
Coq < Open Scope Z_scope.
Coq < Goal forall a b c:Z,
Coq <  (a + b + c)^2 =
Coq <  a * a + b^2 + c * c + 2 * a * b + 2 * a * c + 2 * b * c.
1 subgoal

============================
forall a b c : Z,
(a + b + c) ^ 2 =
  a * a + b ^ 2 + c * c + 2 * a * b + 2 * a * c + 2 * b * c
Coq < intros; ring.
Proof completed.

Coq < Goal forall a b:Z, 2*a*b = 30 ->
Coq <  (a+b)^2 = a^2 + b^2 + 30.
1 subgoal

============================
forall a b : Z, 2 * a * b = 30 -> (a + b) ^ 2 = a ^ 2 + b ^ 2 + 30
Coq < intros a b H; ring [H].
Proof completed.

Variants:
1. ring [term1 ... term_n] decides the equality of two terms modulo ring operations and
rewriting of the equalities defined by term1 ... term_n. Each of term1 ... term_n has to be a
proof of some equality m = p, where m is a monomial (after “abstraction”), p a polynomial and
= the corresponding equality of the ring structure.

2. ring_simplify [term1 ... term_n] t1...tm in ident performs the simplification in
the hypothesis named ident.

Warning: ring_simplify term1; ring_simplify term2 is not equivalent to
ring_simplify term1 term2. In the latter case the variables map is shared between the two
terms, and common subterm t of term1 and term2 will have the same associated variable number. So
the first alternative should be avoided for terms belonging to the same ring theory.

Error messages:
1. not a valid ring equation The conclusion of the goal is not provable in the corre-
sponding ring theory.

2. arguments of ring_simplify do not have all the same type
ring_simplify cannot simplify terms of several rings at the same time. Invoke the
tactic once per ring structure.

3. cannot find a declared ring structure over term No ring has been declared
for the type of the terms to be simplified. Use Add Ring first.

4. cannot find a declared ring structure for equality term Same as
above is the case of the ring tactic.
23.5 Adding a ring structure

Declaring a new ring consists in proving that a ring signature (a carrier set, an equality, and ring operations: `Ring_theory.ring_theory` and `Ring_theory.semi_ring_theory`) satisfies the ring axioms. Semi-rings (rings without \(+\) inverse) are also supported. The equality can be either Leibniz equality, or any relation declared as a setoid (see 24.7). The definition of ring and semi-rings (see module `Ring_theory`) is:

```coq
Record ring_theory : Prop := mk_rt { 
  Radd_0_l : forall x, 0 + x == x;
  Radd_sym : forall x y, x + y == y + x;
  Radd_assoc : forall x y z, x + (y + z) == (x + y) + z;
  Rmul_1_l : forall x, 1 * x == x;
  Rmul_sym : forall x y, x * y == y * x;
  Rmul_assoc : forall x y z, x * (y * z) == (x * y) * z;
  Rdistr_l : forall x y z, (x + y) * z == (x * z) + (y * z);
  Rsub_def : forall x y, x - y == x + -y;
  Ropp_def : forall x, x + (- x) == 0 
}. 

Record semi_ring_theory : Prop := mk_srt { 
  SRadd_0_l : forall n, 0 + n == n;
  SRadd_sym : forall n m, n + m == m + n ;
  SRadd_assoc : forall n m p, n + (m + p) == (n + m) + p;
  SRmul_1_l : forall n, 1 *n == n;
  SRmul_0_l : forall n, 0 *n == 0;
  SRmul_sym : forall n m, n * m == m * n;
  SRmul_assoc : forall n m p, n*(m*p) == (n*m)*p;
  SRdistr_l : forall n m p, (n + m)*p == n*p + m*p 
}. 

This implementation of ring also features a notion of constant that can be parameterized. This can be used to improve the handling of closed expressions when operations are effective. It consists in introducing a type of coefficients and an implementation of the ring operations, and a morphism from the coefficient type to the ring carrier type. The morphism needs not be injective, nor surjective. As an example, one can consider the real numbers. The set of coefficients could be the rational numbers, upon which the ring operations can be implemented. The fact that there exists a morphism is defined by the following properties:

```coq
Record ring_morph : Prop := mkmorph { 
  morph0 : [cO] == 0;
  morph1 : [cI] == 1;
  morph_add : forall x y, [x +! y] == [x]+[y];
  morph_sub : forall x y, [x -! y] == [x]-[y];
  morph_mul : forall x y, [x *! y] == [x]*[y];
  morph_opp : forall x, [-!x] == -[x];
  morph_eq : forall x y, x=?!y = true -> [x] == [y] 
}.
```
Record semi_morph : Prop := mkRmorph {
  Smorph0 : [cO] == 0;
  Smorph1 : [cI] == 1;
  Smorph_add : forall x y, [x ++ y] == [x] + [y];
  Smorph_mul : forall x y, [x *! y] == [x] * [y];
  Smorph_eq : forall x y, x?=!y = true -> [x] == [y]
}. where \(cO\) and \(cI\) denote the 0 and 1 of the coefficient set, \(+!, \times!, \neg!\) are the implementations of the ring operations, \(==\) is the equality of the coefficients, \(?=!\) is an implementation of this equality, and [x] is a notation for the image of \(x\) by the ring morphism.

Since \(\mathbb{Z}\) is an initial ring (and \(\mathbb{N}\) is an initial semi-ring), it can always be considered as a set of coefficients. There are basically three kinds of (semi-)rings:

**abstract rings** to be used when operations are not effective. The set of coefficients is \(\mathbb{Z}\) (or \(\mathbb{N}\) for semi-rings).

**computational rings** to be used when operations are effective. The set of coefficients is the ring itself. The user only has to provide an implementation for the equality.

**customized ring** for other cases. The user has to provide the coefficient set and the morphism.

This implementation of ring can also recognize simple power expressions as ring expressions. A power function is specified by the following property:

Section POWER.
Variable Cpow : Set.
Variable Cp_phi : N -> Cpow.
Variable rpow : R -> Cpow -> R.

Record power_theory : Prop := mkpow_th {
  rpow_pow_N : forall r n, req (rpow r (Cp_phi n)) (pow_N rI rmul r n)
}.
End POWER.

The syntax for adding a new ring is Add Ring name : ring (mod1, ..., mod2). The name is not relevant. It is just used for error messages. The term ring is a proof that the ring signature satisfies the (semi-)ring axioms. The optional list of modifiers is used to tailor the behavior of the tactic. The following list describes their syntax and effects:

**abstract** declares the ring as abstract. This is the default.

**decidable term** declares the ring as computational. The expression term is the correctness proof of an equality test \(?=!\) (which should be evaluable). Its type should be of the form \(forall x y, x?=!y = true \rightarrow x == y\).

**morphism term** declares the ring as a customized one. The expression term is a proof that there exists a morphism between a set of coefficient and the ring carrier (see Ring_theory.ring_morph and Ring_theory.semi_morph).
The ring and field tactic families

setoid \( \text{term}_1 \) \( \text{term}_2 \) forces the use of given setoid. The expression \( \text{term}_1 \) is a proof that the equality is indeed a setoid (see Setoid.Setoid_Theory), and \( \text{term}_2 \) a proof that the ring operations are morphisms (see Ring_theory.ring_eq_ext and Ring_theory.sring_eq_ext). This modifier needs not be used if the setoid and morphisms have been declared.

constants \([L_{\text{tac}}]\) specifies a tactic expression that, given a term, returns either an object of the coefficient set that is mapped to the expression via the morphism, or returns InitialRing.NotConstant. The default behaviour is to map only 0 and 1 to their counterpart in the coefficient set. This is generally not desirable for non trivial computational rings.

preprocess \([L_{\text{tac}}]\) specifies a tactic that is applied as a preliminary step for ring and ring_simplify. It can be used to transform a goal so that it is better recognized. For instance, \(S \ n\) can be changed to \(\text{plus \ 1 \ n}\).

postprocess \([L_{\text{tac}}]\) specifies a tactic that is applied as a final step for ring_simplify. For instance, it can be used to undo modifications of the preprocessor.

power_tac \(\text{term}\) \([L_{\text{tac}}]\) allows ring and ring_simplify to recognize power expressions with a constant positive integer exponent (example: \(x^2\)). The term \(\text{term}\) is a proof that a given power function satisfies the specification of a power function (term has to be a proof of Ring_theory.power_theory) and \(L_{\text{tac}}\) specifies a tactic expression that, given a term, “abstracts” it into an object of type \(N\) whose interpretation via \(Cp_{\phi}\) (the evaluation function of power coefficient) is the original term, or returns InitialRing.NotConstant if not a constant coefficient (i.e. \(L_{\text{tac}}\) is the inverse function of \(Cp_{\phi}\)). See files contrib/setoid_ring/ZArithRing.v and contrib/setoid_ring/RealField.v for examples. By default the tactic does not recognize power expressions as ring expressions.

sign \(\text{term}\) allows ring_simplify to use a minus operation when outputing its normal form, i.e writing \(x - y\) instead of \(x + (-y)\). The term \(\text{term}\) is a proof that a given sign function indicates expressions that are signed (term has to be a proof of Ring_theory.get_sign). See contrib/setoid_ring/IntialRing.v for examples of sign function.

div \(\text{term}\) allows ring and ring_simplify to use moniomals with coefficient other than 1 in the rewriting. The term \(\text{term}\) is a proof that a given division function satisfies the specification of an euclidean division function (term has to be a proof of Ring_theory.div_theory). For example, this function is called when trying to rewrite \(7x\) by \(2x = z\) to tell that \(7 = 3 \times 2 + 1\). See contrib/setoid_ring/IntialRing.v for examples of div function.

Error messages:

1. bad ring structure The proof of the ring structure provided is not of the expected type.

2. bad lemma for decidability of equality The equality function provided in the case of a computational ring has not the expected type.

3. ring operation should be declared as a morphism A setoid associated to the carrier of the ring structure as been found, but the ring operation should be declared as morphism. See 24.7.
23.6 How does it work?

The code of ring is a good example of tactic written using reflection. What is reflection? Basically, it is writing Coq tactics in Coq, rather than in Objective Caml. From the philosophical point of view, it is using the ability of the Calculus of Constructions to speak and reason about itself. For the ring tactic we used Coq as a programming language and also as a proof environment to build a tactic and to prove it correctness.

The interested reader is strongly advised to have a look at the file Ring_polynom.v. Here a type for polynomials is defined:

```coq
Inductive PExpr : Type :=
| PEc : C -> PExpr
| PEX : positive -> PExpr
| PEadd : PExpr -> PExpr -> PExpr
| PESub : PExpr -> PExpr -> PExpr
| PEMul : PExpr -> PExpr -> PExpr
| PEopp : PExpr -> PExpr
| PEpow : PExpr -> N -> PExpr.
```

Polynomials in normal form are defined as:

```coq
Inductive Pol : Type :=
| Pc : C -> Pol
| Pinj : positive -> Pol -> Pol
| PX : Pol -> positive -> Pol -> Pol.
```

where `Pinj n P` denotes `P` in which `Vi` is replaced by `Vi+n`, and `PX P n Q` denotes `P ⊗ V_i^n ⊕ Q'`, `Q'` being `Q` where `Vi` is replaced by `Vi+1`.

Variables maps are represented by list of ring elements, and two interpretation functions, one that maps a variables map and a polynomial to an element of the concrete ring, and the second one that does the same for normal forms:

```coq
Definition PEeval : list R -> PExpr -> R := [...].
Definition Pphi_dev : list R -> Pol -> R := [...].
```

A function to normalize polynomials is defined, and the big theorem is its correctness w.r.t interpretation, that is:

```coq
Definition norm : PExpr -> Pol := [...].
Lemma Pphi_dev_ok :
  forall l pe npe, norm pe = npe -> PEeval l pe == Pphi_dev l npe.
```

So now, what is the scheme for a normalization proof? Let `p` be the polynomial expression that the user wants to normalize. First a little piece of ML code guesses the type of `p`, the ring theory `T` to use, an abstract polynomial `ap` and a variables map `v` such that `p` is `βδι`-equivalent to `(PEeval v ap)`. Then we replace it by `(Pphi_dev v (norm ap))`, using the main correctness theorem and we reduce it to a concrete expression `p'`, which is the concrete normal form of `p`. This is summarized in this diagram:

```
p →βδι (PEeval v ap)
```

```
= (by the main correctness theorem)
```

```
p' ←βδι (Pphi_dev v (norm ap))
```

The user do not see the right part of the diagram. From outside, the tactic behaves like a `βδι` simplification extended with AC rewriting rules. Basically, the proof is only the application of the main correctness theorem to well-chosen arguments.
23.7 Dealing with fields

The field tactic is an extension of the ring to deal with rational expressions. Given a rational expression $F = 0$. It first reduces the expression $F$ to a common denominator $N/D = 0$ where $N$ and $D$ are two ring expressions. For example, if we take $F = (1 - 1/x)x - x + 1$, this gives $N = (x - 1)x - x^2 + x$ and $D = x$. It then calls ring to solve $N = 0$. Note that field also generates non-zero conditions for all the denominators it encounters in the reduction. In our example, it generates the condition $x \neq 0$. These conditions appear as one subgoal which is a conjunction if there are several denominators. Non-zero conditions are always polynomial expressions. For example when reducing the expression $1/(1 + 1/x)$, two side conditions are generated: $x \neq 0$ and $x + 1 \neq 0$. Factorized expressions are broken since a field is an integral domain, and when the equality test on coefficients is complete w.r.t. the equality of the target field, constants can be proven different from zero automatically.

The tactic must be loaded by Require Import Field. New field structures can be declared to the system with the Add Field command (see below). The field of real numbers is defined in module RealField (in textttcontrib/setoid_ring). It is exported by module Rbase, so that requiring Rbase or Reals is enough to use the field tactics on real numbers. Rational numbers in canonical form are also declared as a field in module Qcanon.

Example:

```
Coq < Require Import Reals.
Coq < Open Scope R_scope.
Coq < Goal forall x, x <> 0 ->
Coq < (1 - 1/x) * x - x + 1 = 0.
1 subgoal

=============
forall x : R, x <> 0 -> (1 - 1 / x) * x - x + 1 = 0
```

```Coq
Coq < intros; field; auto.
Proof completed.
```

```
Coq < Goal forall x y, y <> 0 -> y = x -> x/y = 1.
1 subgoal

=============
forall x y : R, y <> 0 -> y = x -> x / y = 1
```

```Coq
Coq < intros x y H H1; field [H1]; auto.
Proof completed.
```

Variants:

1. field [term1 ... term_n] decides the equality of two terms modulo field operations and rewriting of the equalities defined by term1 ... term_n. Each of term1 ... term_n has to be a proof of some equality $m = p$, where $m$ is a monomial (after “abstraction”), $p$ a polynomial and $=$ the corresponding equality of the field structure. Beware that rewriting works with the equality $m = p$ only if $p$ is a polynomial since rewriting is handled by the underlying ring tactic.

2. field_simplify performs the simplification in the conclusion of the goal, $F_1 = F_2$ becomes $N_1/D_1 = N_2/D_2$. A normalization step (the same as the one for rings) is then applied to $N_1$, $D_1$, $N_2$, $D_2$.
$N_2$ and $D_2$. This way, polynomials remain in factorized form during the fraction simplifications. This yields smaller expressions when reducing to the same denominator since common factors can be cancelled.

3. \texttt{field_simplify [term$_1$ ... term$_n$]} performs the simplification in the conclusion of the goal using the equalities defined by term$_1$ ... term$_n$.

4. \texttt{field_simplify [term$_1$ ... term$_n$] t$_1$ ... t$_m$} performs the simplification in the terms t$_1$ ... t$_m$ of the conclusion of the goal using the equalities defined by term$_1$ ... term$_n$.

5. \texttt{field_simplify in H} performs the simplification in the assumption H.

6. \texttt{field_simplify [term$_1$ ... term$_n$] in H} performs the simplification in the assumption H using the equalities defined by term$_1$ ... term$_n$.

7. \texttt{field_simplify [term$_1$ ... term$_n$] t$_1$ ... t$_m$ in H} performs the simplification in the terms t$_1$ ... t$_m$ of the assumption H using the equalities defined by term$_1$ ... term$_n$.

8. \texttt{field_simplify_eq} performs the simplification in the conclusion of the goal removing the denominator. $F_1 = F_2$ becomes $N_1 D_2 = N_2 D_1$.

9. \texttt{field_simplify_eq [term$_1$ ... term$_n$]} performs the simplification in the conclusion of the goal using the equalities defined by term$_1$ ... term$_n$.

10. \texttt{field_simplify_eq in H} performs the simplification in the assumption H.

11. \texttt{field_simplify_eq [term$_1$ ... term$_n$] in H} performs the simplification in the assumption H using the equalities defined by term$_1$ ... term$_n$.

### 23.8 Adding a new field structure

Declaring a new field consists in proving that a field signature (a carrier set, an equality, and field operations: Field\_theory.field\_theory and Field\_theory.semi\_field\_theory) satisfies the field axioms. Semi-fields (fields without $+$ inverse) are also supported. The equality can be either Leibniz equality, or any relation declared as a setoid (see 24.7). The definition of fields and semi-fields is:

```coq
Record field\_theory : Prop := mk\_field {
  F\_R : ring\_theory rO rI radd rmul rsub ropp req;
  F\_l\_neg\_0 : ~ 1 == 0;
  Fdiv\_def : forall p q, p / q == p * / q;
  Finv\_l : forall p, ~ p == 0 -> / p * p == 1
}.

Record semi\_field\_theory : Prop := mk\_sfield {
  SF\_SR : semi\_ring\_theory rO rI radd rmul req;
  SF\_l\_neg\_0 : ~ 1 == 0;
  SFdiv\_def : forall p q, p / q == p * / q;
  SFinv\_l : forall p, ~ p == 0 -> / p * p == 1
}.
```

Coq Reference Manual, V8.2pl2, June 29, 2010
The result of the normalization process is a fraction represented by the following type:

```coq
Record linear : Type := mk_linear {
  num : PExpr C;
  denum : PExpr C;
  condition : list (PExpr C) }.
```

where `num` and `denum` are the numerator and denominator; `condition` is a list of expressions that have appeared as a denominator during the normalization process. These expressions must be proven different from zero for the correctness of the algorithm.

The syntax for adding a new field is `Add Field name : field (mod_1,...,mod_2)`. The name is not relevant. It is just used for error messages. `field` is a proof that the field signature satisfies the (semi-)field axioms. The optional list of modifiers is used to tailor the behaviour of the tactic. Since field tactics are built upon ring tactics, all modifiers of the `Add Ring` apply. There is only one specific modifier:

- **completeness term** allows the field tactic to prove automatically that the image of non-zero coefficients are mapped to non-zero elements of the field. `term` is a proof of `forall x y, [x] = [y] -> x =$! y = true`, which is the completeness of equality on coefficients w.r.t. the field equality.

### 23.9 Legacy implementation

**Warning:** This tactic is the `ring` tactic of previous versions of Coq and it should be considered as deprecated. It will probably be removed in future releases. It has been kept only for compatibility reasons and in order to help moving existing code to the newer implementation described above. For more details, please refer to the Coq Reference Manual, version 8.0.

#### 23.9.1 legacy ring `term_1 ... term_n`

This tactic, written by Samuel Boutin and Patrick Loiseleur, applies associative commutative rewriting on every ring. The tactic must be loaded by `Require Import LegacyRing`. The ring must be declared in the `Add Ring` command. The ring of booleans is predefined; if one wants to use the tactic on `nat` one must first require the module `LegacyArithRing`; for `Z`, do `Require Import LegacyZArithRing`; for `N`, do `Require Import LegacyNArithRing`.

The terms `term_1,...,term_n` must be subterms of the goal conclusion. The tactic `ring` normalizes these terms w.r.t. associativity and commutativity and replace them by their normal form.

**Variants:**

1. `legacy ring` When the goal is an equality `t_1 = t_2`, it acts like `ring_simplify t_1 t_2` and then solves the equality by reflexivity.

2. `ring_nat` is a tactic macro for `repeat rewrite S_to_plus_one; ring`. The theorem `S_to_plus_one` is a proof that `forall (n:nat), S n = plus (S O) n`.

You can have a look at the files `LegacyRing.v, ArithRing.v, ZArithRing.v` to see examples of the `Add Ring` command.
23.9 Legacy implementation

23.9.2 Add a ring structure

It can be done in the Coqtoplevel (No ML file to edit and to link with CoQ). First, ring can handle two kinds of structure: rings and semi-rings. Semi-rings are like rings without an opposite to addition. Their precise specification (in GALLINA) can be found in the file

    contrib/ring/Ring_theory.v

The typical example of ring is \( \mathbb{Z} \), the typical example of semi-ring is \( \text{nat} \).

The specification of a ring is divided in two parts: first the record of constants (\(+\), \(*\), 1, 0, \(\ominus\)) and then the theorems (associativity, commutativity, etc.).

Section Theory_of_semi_rings.

Variable A : Type.
Variable Aone : A.
Variable Azero : A.
(* There is also a "weakly decidable" equality on A. That means that if (A_eq x y)=true then x=y but x=y can arise when (A_eq x y)=false. On an abstract ring the function \( \{x,y:A\}false \) is a good choice. The proof of A_eq_prop is in this case easy. *)

Record Semi_Ring_Theory : Prop :=
{ SR_plus_sym : (n,m:A)[| n + m == m + n |];
SR_plus_assoc : (n,m,p:A)[| n + (m + p) == (n + m) + p |];
SR_mult_sym : (n,m:A)[| n*m == m*n |];
SR_mult_assoc : (n,m,p:A)[| n*(m*p) == (n*m)*p |];
SR_plus_zero_left :(n:A)[| 0 + n == n |];
SR_mult_one_left : (n:A)[| 1*n == n |];
SR_mult_zero_left : (n:A)[| 0*n == 0 |];
SR_distr_left : (n,m,p:A) [| (n + m)*p == n*p + m*p |];
SR_plus_reg_left : (n,m,p:A) [| (n + m) == n + p |] -> m==p;
SR_eq_prop : (x,y:A) (Is_true (Aeq x y)) -> x==y }.

Section Theory_of_rings.

Variable A : Type.
Variable Aone : A.
Variable Azero : A.
Variable Aopp : A -> A.

Record Ring_Theory : Prop :=
{ Th_plus_sym : (n,m:A)[| n + m == m + n |];
The ring and field tactic families

\[
\begin{align*}
\text{Th\_plus\_assoc} &: (n,m,p:A) [\mid n + (m + p) == (n + m) + p ]; \\
\text{Th\_mult\_sym} &: (n,m:A) [\mid n\cdot m == m\cdot n ]; \\
\text{Th\_mult\_assoc} &: (n,m,p:A) [\mid n\cdot (m+p) == (n\cdot m)\cdot p ]; \\
\text{Th\_plus\_zero\_left} &: (n:A) [\mid 0 + n == n ]; \\
\text{Th\_opp\_def} &: (n:A) [\mid n + (-n) == 0 ]; \\
\text{Th\_distr\_left} &: (n,m,p:A) [\mid (n + m)\cdot p == n\cdot p + m\cdot p ]; \\
\text{Th\_eq\_prop} &: (x,y:A) (Is\_true (A\_eq x y)) \rightarrow x == y 
\end{align*}
\]

To define a ring structure on A, you must provide an addition, a multiplication, an opposite function and two unities 0 and 1.

You must then prove all theorems that make (A,Aplus,Amult,Aone,Azero,Aeq) a ring structure, and pack them with the Build_Ring_Theory constructor.

Finally to register a ring the syntax is:

```
Add Legacy Ring A Aplus Amult Aone Azero Ainv Aeq T [ c1 . . . cn ].
```

where A is a term of type Set, Aplus is a term of type A\rightarrow A\rightarrow A, Amult is a term of type A\rightarrow A\rightarrow A, Aone is a term of type A, Azero is a term of type A, Ainv is a term of type A\rightarrow A, Aeq is a term of type A\rightarrow bool, T is a term of type (Ring_Theory A Aplus Amult Aone Azero Ainv Aeq). The arguments c1 ... cn, are the names of constructors which define closed terms: a subterm will be considered as a constant if it is either one of the terms c1 ... cn or the application of one of these terms to closed terms. For nat, the given constructors are S and O, and the closed terms are O, (S O), (S (S O)), ...

Variants:

1. Add Legacy Semi Ring A Aplus Amult Aone Azero Aeq Aeq T [ c1 ... cn ].

   There are two differences with the Add Ring command: there is no inverse function and the term T must be of type (Semi_Ring_Theory A Aplus Amult Aone Azero Aeq).

2. Add Legacy Abstract Ring A Aplus Amult Aone Azero Ainv Aeq T.

   This command should be used for when the operations of rings are not computable; for example the real numbers of theories/REALS/. Here 0 + 1 is not beta-reduced to 1 but you still may want to rewrite it to 1 using the ring axioms. The argument Aeq is not used; a good choice for that function is [x:A]false.

3. Add Legacy Abstract Semi Ring A Aplus Amult Aone Azero Aeq T.

Error messages:

1. Not a valid (semi)ring theory.

   That happens when the typing condition does not hold.

Currently, the hypothesis is made than no more than one ring structure may be declared for a given type in Set or Type. This allows automatic detection of the theory used to achieve the normalization. On popular demand, we can change that and allow several ring structures on the same set.

The table of ring theories is compatible with the COQ sectioning mechanism. If you declare a ring inside a section, the declaration will be thrown away when closing the section. And when you load a compiled file, all the Add Ring commands of this file that are not inside a section will be loaded.
The typical example of ring is \( \mathbb{Z} \), and the typical example of semi-ring is \( \mathbb{N} \). Another ring structure is defined on the booleans.

**Warning:** Only the ring of booleans is loaded by default with the `Ring` module. To load the ring structure for \( \mathbb{N} \), load the module `ArithRing`, and for \( \mathbb{Z} \), load the module `ZArithRing`.

### 23.9.3 legacy field

This tactic written by David Delahaye and Micaela Mayero solves equalities using commutative field theory. Denominators have to be non equal to zero and, as this is not decidable in general, this tactic may generate side conditions requiring some expressions to be non equal to zero. This tactic must be loaded by `Require Import LegacyField`. Field theories are declared (as for legacy ring) with the `Add Legacy Field` command.

### 23.9.4 Add Legacy Field

This vernacular command adds a commutative field theory to the database for the tactic `field`. You must provide this theory as follows:

```coq
Add Legacy Field A Aplus Amult Aone Azero Aopp Aeq Ainv Rth Tinvl
```

where \( A \) is a term of type `Type`, `Aplus` is a term of type \( A \to A \to A \), `Amult` is a term of type \( A \to A \to A \), `Aone` is a term of type \( A \), `Azero` is a term of type \( A \), `Aopp` is a term of type \( A \to A \), `Aeq` is a term of type \( A \to bool \), `Ainv` is a term of type \( A \to A \), `Rth` is a term of type `(Ring_Theory A Aplus Amult Aone Azero Ainv Aeq)`, and `Tinvl` is a term of type `forall n:A, ¬(n=Azero) -> (Amult (Ainv n) n)=Aone`. To build a ring theory, refer to Chapter 23 for more details.

This command adds also an entry in the ring theory table if this theory is not already declared. So, it is useless to keep, for a given type, the `Add Ring` command if you declare a theory with `Add Field`, except if you plan to use specific features of ring (see Chapter 23). However, the module `ring` is not loaded by `Add Field` and you have to make a `Require Import Ring` if you want to call the ring tactic.

**Variants:**

1. `Add Legacy Field A Aplus Amult Aone Azero Aopp Aeq Ainv Rth Tinvl
   with minus:=Aminus`

   Adds also the term `Aminus` which must be a constant expressed by means of `Aopp`.

2. `Add Legacy Field A Aplus Amult Aone Azero Aopp Aeq Ainv Rth Tinvl
   with div:=Adiv`

   Adds also the term `Adiv` which must be a constant expressed by means of `Ainv`.

See also: [42] for more details regarding the implementation of legacy field.

### 23.10 History of ring

First Samuel Boutin designed the tactic `ACDSimpl`. This tactic did lot of rewriting. But the proofs terms generated by rewriting were too big for CoQ’s type-checker. Let us see why:
23 The ring and field tactic families

Coq < Goal forall x y z : Z, x + 3 * y + y * z = x + 3 * y + z * y.
1 subgoal
  =============================
  forall x y z : Z, x + 3 * y + y * z = x + 3 * y + z * y
Coq < intros; rewrite (Zmult_comm y z); reflexivity.
Coq < Save toto.
Coq < Print toto.
toto = fun x y z : Z => eq_ind_r (fun z0 : Z => x + 3 + y + z0 = x + 3 + y + z * y) (refl_equal (x + 3 + y + z * y)) (Zmult_comm y z) : forall x y z : Z, x + 3 + y + y * z = x + 3 + y + z * y
Argument scopes are [Z_scope Z_scope Z_scope]

At each step of rewriting, the whole context is duplicated in the proof term. Then, a tactic that does hundreds of rewriting generates huge proof terms. Since ACDSimpl was too slow, Samuel Boutin rewrote it using reflection (see his article in TACS’97 [18]). Later, the stuff was rewritten by Patrick Loiseleur: the new tactic does not any more require ACDSimpl to compile and it makes use of βδι-reduction not only to replace the rewriting steps, but also to achieve the interleaving of computation and reasoning (see 23.11). He also wrote a few ML code for the Add Ring command, that allow to register new rings dynamically.

Proof terms generated by ring are quite small, they are linear in the number of ⊕ and⊗ operations in the normalized terms. Type-checking those terms requires some time because it makes a large use of the conversion rule, but memory requirements are much smaller.

23.11 Discussion

Efficiency is not the only motivation to use reflection here. ring also deals with constants, it rewrites for example the expression 34+2∗x−x+12 to the expected result x+46. For the tactic ACDSimpl, the only constants were 0 and 1. So the expression 34+2∗(x−1)+12 is interpreted as V₀ ⊕ V₁ ⊗ (V₂ ⊕ 1) ⊕ V₃, with the variables mapping {V₀ ⇒ 34; V₁ ⇒ 2; V₂ ⇒ x; V₃ ⇒ 12}. Then it is rewritten to 34−x+2∗x+12, very far from the expected result. Here rewriting is not sufficient: you have to do some kind of reduction (some kind of computation) to achieve the normalization.

The tactic ring is not only faster than a classical one: using reflection, we get for free integration of computation and reasoning that would be very complex to implement in the classic fashion.

Is it the ultimate way to write tactics? The answer is: yes and no. The ring tactic uses intensively the conversion rule of pCIC, that is replaces proof by computation the most as it is possible. It can be useful in all situations where a classical tactic generates huge proof terms. Symbolic Processing and Tautologies are in that case. But there are also tactics like auto or linear that do many complex computations, using side-effects and backtracking, and generate a small proof term. Clearly, it would be significantly less efficient to replace them by tactics using reflection.

Another idea suggested by Benjamin Werner: reflection could be used to couple an external tool (a rewriting program or a model checker) with Coq. We define (in Coq) a type of terms, a type of traces, and prove a correction theorem that states that replaying traces is safe w.r.t some interpretation. Then we let the external tool do every computation (using side-effects, backtracking, exception, or others features that are not available in pure lambda calculus) to produce the trace: now we can check in Coq that the trace has the expected semantic by applying the correction lemma.
Chapter 24

User defined equalities and relations

Matthieu Sozeau

This chapter presents the extension of several equality related tactics to work over user-defined structures (called setoids) that are equipped with ad-hoc equivalence relations meant to behave as equalities. Actually, the tactics have also been generalized to relations weaker than equivalences (e.g. rewriting systems).

This documentation is adapted from the previous setoid documentation by Claudio Sacerdoti Coen (based on previous work by Clément Renard). The new implementation is a drop-in replacement for the old one \(^1\), hence most of the documentation still applies.

The work is a complete rewrite of the previous implementation, based on the type class infrastructure. It also improves on and generalizes the previous implementation in several ways:

- User-extensible algorithm. The algorithm is separated in two parts: generations of the rewriting constraints (done in ML) and solving of these constraints using type class resolution. As type class resolution is extensible using tactics, this allows users to define general ways to solve morphism constraints.

- Sub-relations. An example extension to the base algorithm is the ability to define one relation as a subrelation of another so that morphism declarations on one relation can be used automatically for the other. This is done purely using tactics and type class search.

- Rewriting under binders. It is possible to rewrite under binders in the new implementation, if one provides the proper morphisms. Again, most of the work is handled in the tactics.

- First-class morphisms and signatures. Signatures and morphisms are ordinary Coq terms, hence they can be manipulated inside Coq, put inside structures and lemmas about them can be proved inside the system. Higher-order morphisms are also allowed.

- Performance. The implementation is based on a depth-first search for the first solution to a set of constraints which can be as fast as linear in the size of the term, and the size of the proof term is linear in the size of the original term. Besides, the extensibility allows the user to customize the proof-search if necessary.

\(^1\)Nicolas Tabareau helped with the glueing
24.1 Relations and morphisms

A parametric relation \( R \) is any term of type \( \forall (x_1 : T_1) \ldots (x_n : T_n), \text{relation } A \). The expression \( A \), which depends on \( x_1 \ldots x_n \), is called the carrier of the relation and \( R \) is said to be a relation over \( A \); the list \( x_1, \ldots, x_n \) is the (possibly empty) list of parameters of the relation.

Example 1 (Parametric relation) It is possible to implement finite sets of elements of type \( A \) as unordered list of elements of type \( A \). The function \( \text{set_eq} \) : \( \forall (A : \text{Type}), \text{relation (list } A \text{) satisfied by two lists with the same elements is a parametric relation over (list } A \text{) with one parameter } A \). The type of \( \text{set_eq} \) is convertible with \( \forall (A : \text{Type}), \text{list } A \rightarrow \text{list } A \rightarrow \text{Prop} \).

An instance of a parametric relation \( R \) with \( n \) parameters is any term \( (R \ t_1 \ldots t_n) \).

Let \( R \) be a relation over \( A \) with \( n \) parameters. A term is a parametric proof of reflexivity for \( R \) if it has type \( \forall (x_1 : T_1) \ldots (x_n : T_n), \text{reflexive } (R \ x_1 \ldots x_n) \). Similar definitions are given for parametric proofs of symmetry and transitivity.

Example 2 (Parametric relation (cont.)) The \( \text{set_eq} \) relation of the previous example can be proved to be reflexive, symmetric and transitive.

A parametric unary function \( f \) of type \( \forall (x_1 : T_1) \ldots (x_n : T_n), A_1 \rightarrow A_2 \) covariantly respects two parametric relation instances \( R_1 \) and \( R_2 \) if, whenever \( x, y \) satisfy \( R_1 \ x \ y \), their images \( f \ x \) and \( f \ y \) satisfy \( R_2 \ (f \ x) \ (f \ y) \). An \( f \) that respects its input and output relations will be called a unary covariant morphism. We can also say that \( f \) is a monotone function with respect to \( R_1 \) and \( R_2 \). The sequence \( x_1, \ldots, x_n \) represents the parameters of the morphism.

Let \( R_1 \) and \( R_2 \) be two parametric relations. The signature of a parametric morphism of type \( \forall (x_1 : T_1) \ldots (x_n : T_n), A_1 \rightarrow A_2 \) that covariantly respects two instances \( I_{R_1} \) and \( I_{R_2} \) of \( R_1 \) and \( R_2 \) is written \( I_{R_1} \leftrightarrow I_{R_2} \). Notice that the special arrow \( \leftrightarrow \), which reminds the reader of covariance, is placed between the two relation instances, not between the two carriers. The signature relation instances and morphism will be typed in a context introducing variables for the parameters.

The previous definitions are extended straightforwardly to \( n \)-ary morphisms, that are required to be simultaneously monotone on every argument.

Morphisms can also be contravariant in one or more of their arguments. A morphism is contravariant on an argument associated to the relation instance \( R \) if it is covariant on the same argument when the inverse relation \( R^{-1} \) (inverse \( R \) in Coq) is considered. The special arrow \( \rightarrow \) is used in signatures for contravariant morphisms.

Functions having arguments related by symmetric relations instances are both covariant and contravariant in those arguments. The special arrow \( \Rightarrow \) is used in signatures for morphisms that are both covariant and contravariant.

An instance of a parametric morphism \( f \) with \( n \) parameters is any term \( f \ t_1 \ldots t_n \).

Example 3 (Morphisms) Continuing the previous example, let \( \text{union} \) : \( \forall (A : \text{Type}), \text{list } A \rightarrow \text{list } A \rightarrow \text{list } A \) perform the union of two sets by appending one list to the other. \( \text{union} \) is a binary morphism parametric over \( A \) that respects the relation instance \( \text{set_eq } A \).

The latter condition is proved by showing \( \forall (A : \text{Type}) \ (S_1 S_1' S_2 S_2') \ (\text{list } A), \text{set_eq } A S_1 S_1' \rightarrow \text{set_eq } A S_2 S_2' \rightarrow \text{set_eq } A (\text{union } A S_1 S_2) (\text{union } A S_1' S_2') \).

The signature of the function \( \text{union } A \) is \( \text{set_eq } A \Rightarrow \text{set_eq } A \Rightarrow \text{set_eq } A \) for all \( A \).
Example 4 (Contravariant morphism) The division function $R_{\text{div}}: \mathbb{R} \to \mathbb{R} \to \mathbb{R}$ is a morphism of signature $\text{le} \leftrightarrow \text{le} \to \text{le}$ where $\text{le}$ is the usual order relation over real numbers. Notice that division is covariant in its first argument and contravariant in its second argument.

Leibniz equality is a relation and every function is a morphism that respects Leibniz equality. Unfortunately, Leibniz equality is not always the intended equality for a given structure.

In the next section we will describe the commands to register terms as parametric relations and morphisms. Several tactics that deal with equality in Coq can also work with the registered relations. The exact list of tactic will be given in Sect. 24.7. For instance, the tactic reflexivity can be used to close a goal $R n n$ whenever $R$ is an instance of a registered reflexive relation. However, the tactics that replace in a context $C[]$ one term with another one related by $R$ must verify that $C[]$ is a morphism that respects the intended relation. Currently the verification consists in checking whether $C[]$ is a syntactic composition of morphism instances that respects some obvious compatibility constraints.

Example 5 (Rewriting) Continuing the previous examples, suppose that the user must prove $\text{set}_{\text{eq}} \text{int} (\text{union} \text{int} (\text{union} \text{int} S1 S2) S2) (f S1 S2)$ under the hypothesis $H: \text{set}_{\text{eq}} \text{int} S2 (\text{nil} \text{int})$. It is possible to use the rewrite tactic to replace the first two occurrences of $S2$ with $\text{nil int}$ in the goal since the context $\text{set}_{\text{eq}} \text{int} (\text{union} \text{int} S1 \text{nil}) \text{nil}$ ($f S1 S2$), being a composition of morphisms instances, is a morphism. However the tactic will fail replacing the third occurrence of $S2$ unless $f$ has also been declared as a morphism.

24.2 Adding new relations and morphisms

A parametric relation $A_{eq}$: $\forall (y_1 : \beta_1 \ldots y_m : \beta_m), \text{relation } (A t_1 \ldots t_n)$ over $(A : \alpha_1 \to \ldots \alpha_n \to \text{Type})$ can be declared with the following command:

```
Add Parametric Relation $(x_1 : T_1) \ldots (x_n : T_k) : (A t_1 \ldots t_n) (A_{eq} t_1' \ldots t_m')$
[reflexivity proved by refl]
[symmetry proved by sym]
[transitivity proved by trans]
as id.
```

after having required the Setoid module with the Require Setoid command.

The identifier id gives a unique name to the morphism and it is used by the command to generate fresh names for automatically provided lemmas used internally.

Notice that the carrier and relation parameters may refer to the context of variables introduced at the beginning of the declaration, but the instances need not be made only of variables. Also notice that $A$ is not required to be a term having the same parameters as $A_{eq}$, although that is often the case in practice (this departs from the previous implementation).

In case the carrier and relations are not parametric, one can use the command Add Relation instead, whose syntax is the same except there is no local context.

The proofs of reflexivity, symmetry and transitivity can be omitted if the relation is not an equivalence relation. The proofs must be instances of the corresponding relation definitions: e.g. the proof of reflexivity must have a type convertible to reflexive $(A t_1 \ldots t_n) (A_{eq} t_1' \ldots t_m')$. Each proof may refer to the introduced variables as well.
Example 6 (Parametric relation) For Leibniz equality, we may declare:

```coq
Add Parametric Relation (A : Type) : A (@eq A)
[reflexivity proved by @refl_equal A]
```

Some tactics (reflexivity, symmetry, transitivity) work only on relations that respect the expected properties. The remaining tactics (replace, rewrite and derived tactics such as autorewrite) do not require any properties over the relation. However, they are able to replace terms with related ones only in contexts that are syntactic compositions of parametric morphism instances declared with the following command.

```coq
Add Parametric Morphism (x_1 : T_1) ...(x_k : T_k)
(ft_1 ... t_n)
with signature sig
as id.
Proof
Qed
```

The command declares $f$ as a parametric morphism of signature $\text{sig}$. The identifier $\text{id}$ gives a unique name to the morphism and it is used as the base name of the type class instance definition and as the name of the lemma that proves the well-definedness of the morphism. The parameters of the morphism as well as the signature may refer to the context of variables. The command asks the user to prove interactively that $f$ respects the relations identified from the signature.

Example 7 We start the example by assuming a small theory over homogeneous sets and we declare set equality as a parametric equivalence relation and union of two sets as a parametric morphism.

```coq
Require Export Setoid.
Require Export Relation_Definitions.
Set Implicit Arguments.
Parameter set: Type -> Type.
Parameter empty: forall A, set A.
Parameter union: forall A, set A -> set A -> set A.
Axiom eq_set_refl: forall A, reflexive _ (eq_set (A:=A)).
Axiom eq_set_sym: forall A, symmetric _ (eq_set (A:=A)).
Axiom eq_set_trans: forall A, transitive _ (eq_set (A:=A)).
Axiom empty_neutral: forall A (S: set A), eq_set (union S (empty A)) S.
Axiom union_compat:
forall (A : Type),
  forall x x': set A, eq_set x x' ->
  forall y y': set A, eq_set y y' ->
  eq_set (union x y) (union x' y').
Add Parametric Relation A : (set A) (@eq_set A)
```
24.3 Rewriting and non reflexive relations

Coq < reflexivity proved by (eq_set_refl (A:=A))
Coq < symmetry proved by (eq_set_sym (A:=A))
Coq < transitivity proved by (eq_set_trans (A:=A))
Coq < as eq_set_rel.

Coq < Add Parametric Morphism A : (@union A) with
Coq < signature (@eq_set A) ==> (@eq_set A) ==> (@eq_set A) as union_mor.
Coq < Proof. exact (@union_compat A). Qed.

Is is possible to reduce the burden of specifying parameters using (maximally inserted) implicit arguments. If A is always set as maximally implicit in the previous example, one can write:

Coq < Add Parametric Relation A : (set A) eq_set
Coq < reflexivity proved by eq_set_refl
Coq < symmetry proved by eq_set_sym
Coq < transitivity proved by eq_set_trans
Coq < as eq_set_rel.

Coq < Add Parametric Morphism A : (@union A) with
Coq < signature eq_set ==> eq_set ==> eq_set as union_mor.
Coq < Proof. exact (@union_compat A). Qed.

We proceed now by proving a simple lemma performing a rewrite step and then applying reflexivity, as we would do working with Leibniz equality. Both tactic applications are accepted since the required properties over eq_set and union can be established from the two declarations above.

Coq < Goal forall (S: set nat),
Coq < eq_set (union (union S empty) S) (union S S).
Coq < Proof. intros. rewrite empty_neutral. reflexivity. Qed.

The tables of relations and morphisms are managed by the type class instance mechanism. The behavior on section close is to generalize the instances by the variables of the section (and possibly hypotheses used in the proofs of instance declarations) but not to export them in the rest of the development for proof search. One can use the Existing Instance command to do so outside the section, using the name of the declared morphism suffixed by _Morphism, or use the Global modifier for the corresponding class instance declaration (see §24.6) at definition time. When loading a compiled file or importing a module, all the declarations of this module will be loaded.

24.3 Rewriting and non reflexive relations

To replace only one argument of an n-ary morphism it is necessary to prove that all the other arguments are related to themselves by the respective relation instances.

Example 8 To replace (union S empty) with S in (union (union S empty) S) (union S S) the rewrite tactic must exploit the monotony of union (axiom union_compat in the previous example). Applying union_compat by hand we are left with the goal eq_set (union S S) (union S S).

Coq Reference Manual, V8.2pl2, June 29, 2010
When the relations associated to some arguments are not reflexive, the tactic cannot automatically prove the reflexivity goals, that are left to the user.

Setoids whose relation are partial equivalence relations (PER) are useful to deal with partial functions. Let $R$ be a PER. We say that an element $x$ is defined if $R \ x \ x$. A partial function whose domain comprises all the defined elements only is declared as a morphism that respects $R$. Every time a rewriting step is performed the user must prove that the argument of the morphism is defined.

**Example 9** Let $eq0 \ be \ fun \ x \ y \ => \ x = y \land \ x \neq 0$ (the smaller PER over non zero elements). Division can be declared as a morphism of signature $eq ==> eq0 ===> eq$. Replace $x$ with $y$ in $div \ x \ n = div \ y \ n$ opens the additional goal $eq0 \ n \ n$ that is equivalent to $n=n \land n\neq0$.

### 24.4 Rewriting and non symmetric relations

When the user works up to relations that are not symmetric, it is no longer the case that any covariant morphism argument is also contravariant. As a result it is no longer possible to replace a term with a related one in every context, since the obtained goal implies the previous one if and only if the replacement has been performed in a contravariant position. In a similar way, replacement in an hypothesis can be performed only if the replaced term occurs in a covariant position.

**Example 10 (Covariance and contravariance)** Suppose that division over real numbers has been defined as a morphism of signature $Zdiv: Zlt ++> Zlt --> Zlt$ (i.e. $Zdiv$ is increasing in its first argument, but decreasing on the second one). Let $<$ denotes $Zlt$. Under the hypothesis $H: x < y$ we have $k < x / y -> k < x / x$, but not $k < y / x -> k < x / x$. Dually, under the same hypothesis $k < x / y -> k < y / y$ holds, but $k < y / x -> k < y / y$ does not. Thus, if the current goal is $k < x / x$, it is possible to replace only the second occurrence of $x$ (in contravariant position) with $y$ since the obtained goal must imply the current one. On the contrary, if $k < x / x$ is an hypothesis, it is possible to replace only the first occurrence of $x$ (in covariant position) with $y$ since the current hypothesis must imply the obtained one.

Contrary to the previous implementation, no specific error message will be raised when trying to replace a term that occurs in the wrong position. It will only fail because the rewriting constraints are not satisfiable. However it is possible to use the `at` modifier to specify which occurrences should be rewritten.

As expected, composing morphisms together propagates the variance annotations by switching the variance every time a contravariant position is traversed.

**Example 11** Let us continue the previous example and let us consider the goal $x / (x / x) < k$. The first and third occurrences of $x$ are in a contravariant position, while the second one is in covariant position. More in detail, the second occurrence of $x$ occurs covariantly in $(x / x)$ (since division is covariant in its first argument), and thus contravariantly in $x / (x / x)$ (since division is contravariant in its second argument), and finally covariantly in $x / (x / x) < k$ (since $<$, as every transitive relation, is contravariant in its first argument with respect to the relation itself).

### 24.5 Rewriting in ambiguous setoid contexts

One function can respect several different relations and thus it can be declared as a morphism having multiple signatures.
Example 12 Union over homogeneous lists can be given all the following signatures: \( \text{eq} \Rightarrow \text{eq} \Rightarrow \text{eq} \) \((\text{eq} \text{ being the equality over ordered lists)}\), \( \text{set\_eq} \Rightarrow \text{set\_eq} \Rightarrow \text{set\_eq} \) \((\text{set\_eq} \text{ being the equality over unordered lists up to duplicates)}\), \( \text{multiset\_eq} \Rightarrow \text{multiset\_eq} \Rightarrow \text{multiset\_eq} \) \((\text{multiset\_eq} \text{ being the equality over unordered lists)}\).

To declare multiple signatures for a morphism, repeat the Add Morphism command.

When morphisms have multiple signatures it can be the case that a rewrite request is ambiguous, since it is unclear what relations should be used to perform the rewriting. Contrary to the previous implementation, the tactic will always choose the first possible solution to the set of constraints generated by a rewrite and will not try to find all possible solutions to warn the user about.

24.6 First class setoids and morphisms

The implementation is based on a first-class representation of properties of relations and morphisms as type classes. That is, the various combinations of properties on relations and morphisms are represented as records and instances of theses classes are put in a hint database. For example, the declaration:

Add Parametric Relation \((x_1 : T_1) \ldots (x_n : T_k) : (A t_1 \ldots t_n) (Aeq t'_1 \ldots t'_m)\)
\([\text{reflexivity proved by refl}]\)
\([\text{symmetry proved by sym}]\)
\([\text{transitivity proved by trans}]\)
as \(\text{id}\).

is equivalent to an instance declaration:

Instance \((x_1 : T_1) \ldots (x_n : T_k) \Rightarrow id : \text{@Equivalence} (A t_1 \ldots t_n) (Aeq t'_1 \ldots t'_m):=\)
\([\text{Equivalence\_Reflexive := refl}]\)
\([\text{Equivalence\_Symmetric := sym}]\)
\([\text{Equivalence\_Transitive := trans}]\).

The declaration itself amounts to the definition of an object of the record type \text{Coq.Classes.RelationClasses.Equivalence} and a hint added to the \text{typeclass\_instances} hint database. Morphism declarations are also instances of a type class defined in \text{Classes.Morphisms}. See the documentation on type classes \text{18} and the theories files in \text{Classes} for further explanations.

One can inform the rewrite tactic about morphisms and relations just by using the typeclass mechanism to declare them using Instance and Context vernacular commands. Any object of type Morphism in the local context will also be automatically used by the rewriting tactic to solve constraints.

Other representations of first class setoids and morphisms can also be handled by encoding them as records. In the following example, the projections of the setoid relation and of the morphism function can be registered as parametric relations and morphisms.

Example 13 (First class setoids) Coq < Require Import Relation_Definitions Setoid.

Coq < Record Setoid: Type :=
Coq < { car:Type;
Coq < eq:car->car->Prop;
Coq < refl: reflexive _ eq;
Coq < sym: symmetric _ eq;
Coq < trans: transitive _ eq
Coq < }.
Coq < Add Parametric Relation (s : Setoid) : (@car s) (@eq s)
Coq < reflexivity proved by (refl s)
Coq < symmetry proved by (sym s)
Coq < transitivity proved by (trans s) as eq_rel.
Coq < Record Morphism (S1 S2:Setoid): Type :=
Coq < { f:car S1 ->car S2;
Coq < compat: forall (x1 x2: car S1), eq S1 x1 x2 -> eq S2 (f x1) (f x2) }.
Coq < Add Parametric Morphism (S1 S2 : Setoid) (M : Morphism S1 S2) :
Coq < (@f S1 S2 M) with signature (@eq S1 ==> @eq S2) as apply_mor.
Coq < Proof. apply (compat S1 S2 M). Qed.
Coq < Lemma test: forall (S1 S2:Setoid) (m: Morphism S1 S2)
Coq < (x y: car S1), eq S1 x y -> eq S2 (f _ _ m x) (f _ _ m y).

24.7 Tactics enabled on user provided relations

The following tactics, all prefixed by setoid_, deal with arbitrary registered relations and morphisms. Moreover, all the corresponding unprefixed tactics (i.e. reflexivity, symmetry, transitivity, replace, rewrite) have been extended to fall back to their prefixed counterparts when the relation involved is not Leibniz equality. Notice, however, that using the prefixed tactics it is possible to pass additional arguments such as using relation.

```
setoid_reflexivity
setoid_symmetry [in ident]
setoid_transitivity
setoid_rewrite [orientation] term [at occs] [in ident]
setoid_replace term with term [in ident] [using relation term] [by tactic]
```

The using relation arguments cannot be passed to the unprefixed form. The latter argument tells the tactic what parametric relation should be used to replace the first tactic argument with the second one. If omitted, it defaults to the DefaultRelation instance on the type of the objects. By default, it means the most recent Equivalence instance in the environment, but it can be customized by declaring new DefaultRelation instances. As leiniz equality is a declared equivalence, it will fall back to it if no other relation is declared on a type.

Every derived tactic that is based on the unprefixed forms of the tactics considered above will also work up to user defined relations. For instance, it is possible to register hints for autorewrite that are not proof of Leibniz equalities. In particular it is possible to exploit autorewrite to simulate normalization in a term rewriting system up to user defined equalities.

24.8 Printing relations and morphisms

The Print Instances command can be used to show the list of currently registered Reflexive (using Print Instances Reflexive), Symmetric or Transitive relations, Equivalences, PreOrders, PERs, and Morphisms. When the rewriting tactics refuse to replace
a term in a context because the latter is not a composition of morphisms, the \texttt{Print Instances} commands can be useful to understand what additional morphisms should be registered.

### 24.9 Deprecated syntax and backward incompatibilities

Due to backward compatibility reasons, the following syntax for the declaration of setoids and morphisms is also accepted.

\begin{verbatim}
Add Setoid \texttt{A Aeq ST as ident}
\end{verbatim}

where \texttt{Aeq} is a congruence relation without parameters, \texttt{A} is its carrier and \texttt{ST} is an object of type \texttt{(Setoid_Theory A Aeq)} (i.e. a record packing together the reflexivity, symmetry and transitivity lemmas). Notice that the syntax is not completely backward compatible since the identifier was not required.

\begin{verbatim}
Add Morphism \texttt{f:ident}.
Proof.
  ...
Qed.
\end{verbatim}

The latter command also is restricted to the declaration of morphisms without parameters. It is not fully backward compatible since the property the user is asked to prove is slightly different: for \(n\)-ary morphisms the hypotheses of the property are permuted; moreover, when the morphism returns a proposition, the property is now stated using a bi-implication in place of a simple implication. In practice, porting an old development to the new semantics is usually quite simple.

Notice that several limitations of the old implementation have been lifted. In particular, it is now possible to declare several relations with the same carrier and several signatures for the same morphism. Moreover, it is now also possible to declare several morphisms having the same signature. Finally, the \texttt{replace} and \texttt{rewrite} tactics can be used to replace terms in contexts that were refused by the old implementation. As discussed in the next section, the semantics of the new \texttt{setoid_rewrite} command differs slightly from the old one and \texttt{rewrite}.

### 24.10 Rewriting under binders

\textbf{Warning:} Due to compatibility issues, this feature is enabled only when calling the \texttt{setoid_rewrite} tactics directly and not \texttt{rewrite}.

To be able to rewrite under binding constructs, one must declare morphisms with respect to pointwise (setoid) equivalence of functions. Example of such morphisms are the standard \texttt{all} and \texttt{ex} combinators for universal and existential quantification respectively. They are declared as morphisms in the \texttt{Classes.Morphisms_Prop} module. For example, to declare that universal quantification is a morphism for logical equivalence:

\begin{verbatim}
Coq < Instance all_iff_morphism (A : Type) :
Coq <   Morphism (pointwise_relation A iff ==> iff) (@all A).
Toplevel input, characters 51-83:
  >   Morphism (pointwise_relation A iff ==> iff) (@all A).
  >     ^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^^
Error: Unknown interpretation for notation "\_ ==> \_".
\end{verbatim}
One then has to show that if two predicates are equivalent at every point, their universal quantifications are equivalent. Once we have declared such a morphism, it will be used by the setoid rewriting tactic each time we try to rewrite under an `all` application (products in `Prop` are implicitly translated to such applications).

Indeed, when rewriting under a lambda, binding variable `x`, say from `P x` to `Q x` using the relation `iff`, the tactic will generate a proof of `pointwise_relation A iff (fun x => P x) (fun x => Q x)` from the proof of `iff (P x) (Q x)` and a constraint of the form `Morphism (pointwise_relation A iff ==> ?) m` will be generated for the surrounding morphism `m`.

Hence, one can add higher-order combinators as morphisms by providing signatures using pointwise extension for the relations on the functional arguments (or whatever subrelation of the pointwise extension). For example, one could declare the `map` combinator on lists as a morphism:

```coq
Coq < Instance map_morphism '{Equivalence A eqA, Equivalence B eqB} :
Coq < Morphism ((eqA ==> eqB) ==> list_equiv eqA ==> list_equiv eqB)
Coq < (@map A B).
```

where `list_equiv` implements an equivalence on lists parameterized by an equivalence on the elements.

Note that when one does rewriting with a lemma under a binder using `setoid_rewrite`, the application of the lemma may capture the bound variable, as the semantics are different from rewrite where the lemma is first matched on the whole term. With the new `setoid_rewrite`, matching is done on each subterm separately and in its local environment, and all matches are rewritten simultaneously by default. The semantics of the previous `setoid_rewrite` implementation can almost be recovered using the `at 1` modifier.

### 24.11 Sub-relations

Sub-relations can be used to specify that one relation is included in another, so that morphisms signatures for one can be used for the other. If a signature mentions a relation `R` on the left of an arrow `==>`, then the signature also applies for any relation `S` that is smaller than `R`, and the inverse applies on the right of an arrow. One can then declare only a few morphisms instances that generate the complete set of signatures for a particular constant. By default, the only declared subrelation is `iff`, which is a subrelation of `impl` and `inverse impl` (the dual of implication). That’s why we can declare only two morphisms for `conjunction`: `Morphism (impl ==> impl ==> impl)` and `and Morphism (iff ==> iff ==> iff)` and. This is sufficient to satisfy any rewriting constraints arising from a rewrite using `iff.impl` or `inverse.impl` through `and`.

Sub-relations are implemented in `Classes.Morphisms` and are a prime example of a mostly user-space extension of the algorithm.

### 24.12 Constant unfolding

The resolution tactic is based on type classes and hence regards user-defined constants as transparent by default. This may slow down the resolution due to a lot of unifications (all the declared `Morphism` instances are tried at each node of the search tree). To speed it up, declare your constant as rigid for proof search using the command `Typeclasses Opaque` (see §18.5.4).
Chapter 25

Calling external provers

25.1 The gappa tactic

Sylvie Boldo, Guillaume Melquiond, Jean-Christophe Filliâtre

The **gappa** tactic invokes the Gappa tool\(^1\) to solve properties about floating-point or fixed-point arithmetic. It can also solve simple inequalities over real numbers.

The Gappa tool must be installed and its executable (called **gappa**) must be in the user program path. The Coq support library for Gappa must also be installed (it is available from Gappa’s web site). This library provides a `Gappa_tactic` module, which must be loaded for the tactic to work properly.

The **gappa** tactic only handles goals and hypotheses that are double inequalities \(f_1 \leq e \leq f_2\) where \(f_1\) and \(f_2\) are dyadic constants and \(e\) a real-valued expression. Here is an example of a goal solved by **gappa**:

```
Lemma test_gappa :
  forall x y:R,
  3/4 <= x <= 3 ->
  0 <= sqrt x <= 1775 * (powerRZ 2 (-10)).
Proof.
gappa.
Qed.
```

Gappa supports floating-point rounding operations (as functions over real numbers). Here is an example involving double-precision floating-point numbers with rounding toward zero:

```
Definition rnd := gappa_rounding (rounding_float roundZR 53 1074).

Lemma test_gappa2 :
  forall a_ b_ a b : R,
  a = rnd a_ ->
  b = rnd b_ ->
  52 / 16 <= a <= 53 / 16 ->
```

\(^1\) [http://lipforge.ens-lyon.fr/www/gappa/](http://lipforge.ens-lyon.fr/www/gappa/)
22 / 16 <= b <= 30 / 16 ->
0 <= \text{rnd} (a - b) - (a - b) <= 0.

Proof.
   unfold \text{rnd}; \text{gappa}.
Qed.

The function \text{gappa\_rounding} declares a rounding mode recognized by the \text{gappa} tactic. Rounding modes are built using constants such as \text{rounding\_float} and \text{roundZR} provided by the Gappa support library.
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Global Index

||, 215
*, 85, 91
+, 85, 91
-, 91
/, 91
;[...|...|...], 212
<, 91
<=, 91
>, 91
>=, 91
?, 152
?=, 91
%, 288
&+, 86
_., 34
(A)+{B}, 86
{x:A & (P x)}, 86
{x:A | (P x)}, 86
hyperpage, 86
2-level approach, 194

Add Legacy Abstract Semi Ring, 392
Add Legacy Field, 393
Add Legacy Ring, 390, 392
Add Legacy Semi Ring, 390, 392
Add LoadPath, 136
Add ML Path, 137
Add Morphism, 403
Add Parametric Morphism, 398
Add Parametric Relation, 397
Add Printing If ident, 57
Add Printing Let ident, 57
Add Rec LoadPath, 136
Add Rec ML Path, 137
Add Relation, 397
Add Ring, 200, 384
Add Setoid, 403
admit, 163
Admit Obligations, 378
Admitted, 49, 144
call, 83
and, 83
and_rect, 87
app, 94
applications, 34
apply, 156
apply ... with, 156
apply ... in, 159
Arguments Scope, 288
Arithmetical notations, 91
Arity, 105
assert, 158
assert as, 159
assert by, 159
Associativity, 280
assumption, 152
auto, 195
autorewrite, 201
Axiom, 38
Axiom (and coercions), 342
Back, 138
BackTo, 139
Backtrack, 138
Bad Magic Number, 135
β-reduction, 100
Bind Scope, 289
binders, 32
Binding list, 164
BNF metasyntax, 29
bool, 85
bool_choice, 86
byte-code, 297
C-zar, 251
Calculus of (Co)Inductive Constructions, 95
Canonical Structure, 78
case, 176
case_eq, 176
Cases, 329
Cast, 34
cbv, 166
Cd, 136
change, 162
change ... in, 162
Check, 128
Choice, 86
Choice2, 86
CIC, 95
Class, 353
classical_left, 195
classical_right, 195
Clauses, 166
clear, 153
clearbody, 153
Close Scope, 288
Coercion, 80, 341, 342
Coercions, 80
and records, 344
and sections, 344
classes, 339
Funclass, 340
identity, 340
inheritance graph, 341
presentation, 339
Sortclass, 340
cofix, 163
CoFixpoint, 48
CoFixpoint ... where ..., 284
CoInductive, 45
CoInductive (and coercions), 342
Combined Scheme, 209, 226
Comments, 29
compare, 187
Compiled files, 133
Compute, 128
compute, 166
congruence, 198
conj, 83
Conjecture, 38
Connectives, 83
Constant, 39
creator, 170
Context, 98
corollary, 49
Cut, 158
cutrewrite, 184
Datatypes, 84
Debugger, 303
decide equality, 187
Declarations, 38
Declare Implicit Tactic, 207
Declare Left Step, 186
Declare ML Module, 135
Declare Right Step, 186
decompose, 181
decompose record, 182
decompose sum, 182
Defined, 49, 144
Definition, 39, 145
Definitions, 39
Delimit Scope, 288
δ-reduction, 39, 100
Dependencies, 304
dependent destruction, 181
dependent induction, 180
dependent induction ...
generalizing, 181
dependent inversion, 192
dependent inversion ... as, 192
dependent inversion ... as ...
with, 193
dependent inversion ... with, 192
dependent inversion_clear ... as, 192
dependent inversion_clear ...
with, 193
dependent rewrite ->, 190
dependent rewrite <->, 190
Derive Dependent Inversion, 193
Derive Dependent
Inversion_clear, 193
Derive Inversion, 193
Derive Inversion_clear, 193
Derive Inversion_clear ... with, 193
destruct, 174
discriminate, 187
discrR, 93
Disjunctive/conjunctive introduction patterns, 176
do, 214
double induction, 179
Drop, 139
eapply, 156, 224
eapply ... in, 160
eassumption, 152
eauto, 196
ecase, 176
econstructor, 171
edestruct, 175
discriminate, 187
eelim, 174
eexact, 152
eexists, 171
einduction, 173
einjection, 188
eleft, 171
elim ... using, 174
Elimination
   Empty elimination, 112
   Singleton elimination, 112
Elimination sorts, 110
elimtype, 174
Emacs, 316
End, 61, 63, 64
Environment, 39, 98
Environment variables, 298
eq, 84
eq_add_S, 87
eq_ind_r, 84
eq_rec_r, 84
eq_rect, 84, 87
eq_rect_r, 84
eq_S, 87
Equality, 84
erewrite, 184
eright, 171
error, 87
esimplify_eq, 190
esplit, 171
η-conversion, 101
η-reduction, 101
Eval, 128
eval
   in Ltac, 220
evar, 163
ex, 83
ex2, 83
ex_intro, 83
ex_intro2, 83
exact, 152
Example, 39
Exc, 87
exist, 86
exist2, 86
<table>
<thead>
<tr>
<th>Term</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Existential, 146</td>
<td>421</td>
</tr>
<tr>
<td>Existing Instance, 354</td>
<td></td>
</tr>
<tr>
<td>exists, 83, 171</td>
<td></td>
</tr>
<tr>
<td>exists2, 83</td>
<td></td>
</tr>
<tr>
<td>existT, 86</td>
<td></td>
</tr>
<tr>
<td>existT2, 86</td>
<td></td>
</tr>
<tr>
<td>Explicitly given implicit arguments, 76</td>
<td></td>
</tr>
<tr>
<td>Export, 68</td>
<td></td>
</tr>
<tr>
<td>Extract Constant, 366</td>
<td></td>
</tr>
<tr>
<td>Extract Inductive, 367</td>
<td></td>
</tr>
<tr>
<td>Extraction, 363</td>
<td></td>
</tr>
<tr>
<td>Extraction, 128, 363</td>
<td></td>
</tr>
<tr>
<td>Extraction Blacklist, 367</td>
<td></td>
</tr>
<tr>
<td>Extraction Inline, 365</td>
<td></td>
</tr>
<tr>
<td>Extraction Language, 364</td>
<td></td>
</tr>
<tr>
<td>Extraction Module, 363</td>
<td></td>
</tr>
<tr>
<td>Extraction NoInline, 365</td>
<td></td>
</tr>
<tr>
<td>f_equal, 84, 186</td>
<td></td>
</tr>
<tr>
<td>f_equali, 84</td>
<td></td>
</tr>
<tr>
<td>Fact, 49, 145</td>
<td></td>
</tr>
<tr>
<td>fail, 216</td>
<td></td>
</tr>
<tr>
<td>False, 83</td>
<td></td>
</tr>
<tr>
<td>false, 85</td>
<td></td>
</tr>
<tr>
<td>False_rec, 87</td>
<td></td>
</tr>
<tr>
<td>False_rect, 87</td>
<td></td>
</tr>
<tr>
<td>field, 200, 388</td>
<td></td>
</tr>
<tr>
<td>field_simplify, 200, 388</td>
<td></td>
</tr>
<tr>
<td>first, 215</td>
<td></td>
</tr>
<tr>
<td>firstorder, 198</td>
<td></td>
</tr>
<tr>
<td>firstorder using, 198</td>
<td></td>
</tr>
<tr>
<td>firstorder with, 198</td>
<td></td>
</tr>
<tr>
<td>firstorder tactic, 198</td>
<td></td>
</tr>
<tr>
<td>Fix, 113</td>
<td></td>
</tr>
<tr>
<td>fix, 162</td>
<td></td>
</tr>
<tr>
<td>fix ident{ [...], 36</td>
<td></td>
</tr>
<tr>
<td>fix_eq, 89</td>
<td></td>
</tr>
<tr>
<td>Fix_F, 89</td>
<td></td>
</tr>
<tr>
<td>Fix_F_eq, 89</td>
<td></td>
</tr>
<tr>
<td>Fix_F_inv, 89</td>
<td></td>
</tr>
<tr>
<td>Fixpoint, 45</td>
<td></td>
</tr>
<tr>
<td>Fixpoint ... where ..., 284</td>
<td></td>
</tr>
<tr>
<td>flat_map, 94</td>
<td></td>
</tr>
<tr>
<td>Focus, 147</td>
<td></td>
</tr>
<tr>
<td>fold, 169</td>
<td></td>
</tr>
<tr>
<td>fold_left, 94</td>
<td></td>
</tr>
<tr>
<td>fold_right, 94</td>
<td></td>
</tr>
<tr>
<td>form, 31</td>
<td></td>
</tr>
<tr>
<td>fourier, 201</td>
<td></td>
</tr>
<tr>
<td>fresh</td>
<td></td>
</tr>
<tr>
<td>in Ltac, 220</td>
<td></td>
</tr>
<tr>
<td>fst, 85</td>
<td></td>
</tr>
<tr>
<td>fun</td>
<td></td>
</tr>
<tr>
<td>in Ltac, 216</td>
<td></td>
</tr>
<tr>
<td>Function, 58</td>
<td></td>
</tr>
<tr>
<td>functional induction, 182, 227</td>
<td></td>
</tr>
<tr>
<td>Functional Scheme, 209, 227</td>
<td></td>
</tr>
<tr>
<td>Gallina, 29, 51</td>
<td></td>
</tr>
<tr>
<td>gallina, 316</td>
<td></td>
</tr>
<tr>
<td>gappa, 405</td>
<td></td>
</tr>
<tr>
<td>ge, 88</td>
<td></td>
</tr>
<tr>
<td>generalize, 161</td>
<td></td>
</tr>
<tr>
<td>generalize dependent, 161</td>
<td></td>
</tr>
<tr>
<td>Global Implicit Arguments, 73, 74</td>
<td></td>
</tr>
<tr>
<td>Goal, 49, 143</td>
<td></td>
</tr>
<tr>
<td>goal, 151</td>
<td></td>
</tr>
<tr>
<td>Goal clauses, 166</td>
<td></td>
</tr>
<tr>
<td>gt, 88</td>
<td></td>
</tr>
<tr>
<td>Guarded, 148</td>
<td></td>
</tr>
<tr>
<td>head, 94</td>
<td></td>
</tr>
<tr>
<td>Head normal form, 101</td>
<td></td>
</tr>
<tr>
<td>Hint, 202</td>
<td></td>
</tr>
<tr>
<td>Hint Constructors, 203</td>
<td></td>
</tr>
<tr>
<td>Hint Extern, 204</td>
<td></td>
</tr>
<tr>
<td>Hint Immediate, 203</td>
<td></td>
</tr>
<tr>
<td>Hint Opaque, 204</td>
<td></td>
</tr>
<tr>
<td>Hint Resolve, 202</td>
<td></td>
</tr>
<tr>
<td>Hint Rewrite, 206</td>
<td></td>
</tr>
<tr>
<td>Hint Transparent, 204</td>
<td></td>
</tr>
<tr>
<td>Hint Unfold, 204</td>
<td></td>
</tr>
<tr>
<td>Hints databases, 202</td>
<td></td>
</tr>
<tr>
<td>hnf, 167</td>
<td></td>
</tr>
<tr>
<td>Hypotheses, 38</td>
<td></td>
</tr>
<tr>
<td>Hypothesis, 38</td>
<td></td>
</tr>
<tr>
<td>Hypothesis (and coercions), 342</td>
<td></td>
</tr>
<tr>
<td>I, 83</td>
<td></td>
</tr>
<tr>
<td>ident, 29</td>
<td></td>
</tr>
<tr>
<td>identity, 85</td>
<td></td>
</tr>
<tr>
<td>Identity Coercion, 342</td>
<td></td>
</tr>
<tr>
<td>idtac, 215</td>
<td></td>
</tr>
<tr>
<td>if ... then ... else, 54</td>
<td></td>
</tr>
</tbody>
</table>
IF_then_else, 83
iff, 83
Implicit Arguments, 71, 74
Implicit arguments, 70
Implicit Types, 79
Import, 67
Include, 62, 63
induction, 171
Inductive, 40
Inductive (and coercions), 342
Inductive definitions, 40
Inductive ... where ..., 284
Infix, 283
info, 220
injection, 188
injection ... as ..., 190
inl, 85
inleft, 86
Inline, 63
inr, 85
inright, 86
Inspect, 127
Instance, 353
instantiate, 163
integer, 30
Interpretation scopes, 287
intro, 154
intro after, 155
intro at bottom, 155
intro at top, 155
intro before, 155
Introduction patterns, 176
intros, 154
intros intro_pattern, 176
intros until, 155
intuition, 197
inversion, 191, 229
inversion ... as ..., 191
inversion ... as ... in, 192
inversion ... in, 192
inversion ... using, 193
inversion ... using ... in, 193
inversion_clear, 191
inversion_clear ... as ... in, 192
inversion_clear ... in, 192
inversion_clear ... in, 192
\(\iota\)-reduction, 100, 113, 116
IsSucc, 87
\(\lambda\)-calculus, 97
lapply, 157
\LaTeX, 315
lazy, 166
lazymatch
in Ltac, 218
lazymatch goal
in Ltac, 219
lazymatch reverse goal
in Ltac, 219
le, 88
le_n, 88
le_S, 88
left, 86, 171
legacy field, 393
legacy ring, 390
Lemma, 49, 145
length, 94
Let, 40, 145
let
in Ltac, 216
let ... in, 55
let ... in, 55
let rec
in Ltac, 216
let-in, 34
Lexical conventions, 29
Libraries, 68
Load, 133
Load Verbose, 133
Loadpath, 135
Local Coercion, 341, 342
local context, 143
Local definitions, 34
Local Implicit Arguments, 73, 74
Local Strategy, 142
Locate, 131, 284
Locate File, 137
Locate Library, 137
Locate Module, 68
Logical paths, 68
lt, 88
Ltac
eval, 220
<table>
<thead>
<tr>
<th>Term</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>external</td>
<td>221</td>
</tr>
<tr>
<td>fresh</td>
<td>220</td>
</tr>
<tr>
<td>fun</td>
<td>216</td>
</tr>
<tr>
<td>lazymatch</td>
<td>218</td>
</tr>
<tr>
<td>lazymatch goal</td>
<td>219</td>
</tr>
<tr>
<td>lazymatch reverse goal</td>
<td>219</td>
</tr>
<tr>
<td>let</td>
<td>216</td>
</tr>
<tr>
<td>let rec</td>
<td>216</td>
</tr>
<tr>
<td>match</td>
<td>217</td>
</tr>
<tr>
<td>match goal</td>
<td>218</td>
</tr>
<tr>
<td>match reverse goal</td>
<td>218</td>
</tr>
<tr>
<td>type of</td>
<td>220</td>
</tr>
<tr>
<td>Ltac</td>
<td>221</td>
</tr>
<tr>
<td>Makefile</td>
<td>304</td>
</tr>
<tr>
<td>Man pages</td>
<td>316</td>
</tr>
<tr>
<td>map</td>
<td>94</td>
</tr>
<tr>
<td>match</td>
<td></td>
</tr>
<tr>
<td>in Ltac</td>
<td>217</td>
</tr>
<tr>
<td>match...with...end</td>
<td>35, 54, 110</td>
</tr>
<tr>
<td>match goal</td>
<td></td>
</tr>
<tr>
<td>in Ltac</td>
<td>218</td>
</tr>
<tr>
<td>match reverse goal</td>
<td></td>
</tr>
<tr>
<td>in Ltac</td>
<td>218</td>
</tr>
<tr>
<td>ML-like patterns</td>
<td>54, 329</td>
</tr>
<tr>
<td>mod</td>
<td>91</td>
</tr>
<tr>
<td>Module</td>
<td>62, 63</td>
</tr>
<tr>
<td>Module Type</td>
<td>63</td>
</tr>
<tr>
<td>Modules</td>
<td>61</td>
</tr>
<tr>
<td>move</td>
<td>153</td>
</tr>
<tr>
<td>mult</td>
<td>87</td>
</tr>
<tr>
<td>mult_n_O</td>
<td>87</td>
</tr>
<tr>
<td>mult_n_Sm</td>
<td>87</td>
</tr>
<tr>
<td>Mutual Inductive</td>
<td>43</td>
</tr>
<tr>
<td>n_Sn</td>
<td>87</td>
</tr>
<tr>
<td>Naming introduction patterns</td>
<td>176</td>
</tr>
<tr>
<td>nat</td>
<td>85</td>
</tr>
<tr>
<td>nat_case</td>
<td>88</td>
</tr>
<tr>
<td>nat_double_ind</td>
<td>88</td>
</tr>
<tr>
<td>nat_scope</td>
<td>91</td>
</tr>
<tr>
<td>native code</td>
<td>297</td>
</tr>
<tr>
<td>Next Obligation</td>
<td>378</td>
</tr>
<tr>
<td>None</td>
<td>85</td>
</tr>
<tr>
<td>Normal form</td>
<td>101</td>
</tr>
<tr>
<td>not</td>
<td>83</td>
</tr>
<tr>
<td>not_eq_S</td>
<td>87</td>
</tr>
<tr>
<td>Notation</td>
<td>279, 292</td>
</tr>
<tr>
<td>Notations for lists</td>
<td>94</td>
</tr>
<tr>
<td>Notations for real numbers</td>
<td>92</td>
</tr>
<tr>
<td>notT</td>
<td>90</td>
</tr>
<tr>
<td>nth</td>
<td>94</td>
</tr>
<tr>
<td>num</td>
<td>30</td>
</tr>
<tr>
<td>O</td>
<td>85</td>
</tr>
<tr>
<td>O_S</td>
<td>87</td>
</tr>
<tr>
<td>Obligation</td>
<td>378</td>
</tr>
<tr>
<td>Obligation Tactic</td>
<td>378</td>
</tr>
<tr>
<td>Obligations</td>
<td>378</td>
</tr>
<tr>
<td>Occurrences clauses</td>
<td>164</td>
</tr>
<tr>
<td>omega</td>
<td>200, 355</td>
</tr>
<tr>
<td>Opaque</td>
<td>141</td>
</tr>
<tr>
<td>Open Scope</td>
<td>288</td>
</tr>
<tr>
<td>option</td>
<td>85</td>
</tr>
<tr>
<td>Options of the command line</td>
<td>298</td>
</tr>
<tr>
<td>or</td>
<td>83</td>
</tr>
<tr>
<td>or_introl</td>
<td>83</td>
</tr>
<tr>
<td>or_intror</td>
<td>83</td>
</tr>
<tr>
<td>pair</td>
<td>85</td>
</tr>
<tr>
<td>pairT</td>
<td>90</td>
</tr>
<tr>
<td>Parameter</td>
<td>38</td>
</tr>
<tr>
<td>Parameter (and coercions)</td>
<td>342</td>
</tr>
<tr>
<td>Parameters</td>
<td>38</td>
</tr>
<tr>
<td>pattern</td>
<td>169</td>
</tr>
<tr>
<td>pCIC</td>
<td>95</td>
</tr>
<tr>
<td>Peano’s arithmetic</td>
<td>91</td>
</tr>
<tr>
<td>Physical paths</td>
<td>68</td>
</tr>
<tr>
<td>plus</td>
<td>87</td>
</tr>
<tr>
<td>plus_n_O</td>
<td>87</td>
</tr>
<tr>
<td>plus_n_Sm</td>
<td>87</td>
</tr>
<tr>
<td>pose</td>
<td>157</td>
</tr>
<tr>
<td>pose proof</td>
<td>159</td>
</tr>
<tr>
<td>Positivity</td>
<td>105</td>
</tr>
<tr>
<td>Precedences</td>
<td>280</td>
</tr>
<tr>
<td>pred</td>
<td>87</td>
</tr>
<tr>
<td>pred_Sn</td>
<td>87</td>
</tr>
<tr>
<td>Predicative Calculus of (Co)Inductive Constructions</td>
<td>95</td>
</tr>
<tr>
<td>Preterm</td>
<td>378</td>
</tr>
<tr>
<td>Print</td>
<td>127</td>
</tr>
<tr>
<td>Print All</td>
<td>127</td>
</tr>
<tr>
<td>Print Assumptions</td>
<td>128</td>
</tr>
<tr>
<td>Print Canonical Projections</td>
<td>79</td>
</tr>
<tr>
<td>Print Classes</td>
<td>343</td>
</tr>
<tr>
<td>Print Coercion Paths</td>
<td>343</td>
</tr>
</tbody>
</table>
Print Coercions, 343
Print Extraction Inline, 365
Print Grammar constr, 281
Print Grammar pattern, 281
Print Graph, 343
Print Hint, 206
Print HintDb, 206
Print Implicit, 77
Print Libraries, 135
Print LoadPath, 137
Print Ltac, 222
Print ML Modules, 135
Print ML Path, 137
Print Module, 68
Print Module Type, 68
Print Scope, 292
Print Scopes, 292
Print Section, 127
Print Table Printing If, 57
Print Table Printing Let, 57
Print Term, 127
Print Universes, 80
Print Visibility, 291
Print XML, 313
prod, 85
prodT, 90
products, 34
Program, 375
Program Definition, 376
Program Fixpoint, 377
Program Instance, 350, 354
Program Lemma, 378
Programming, 85
progress, 215
proj1, 83
proj2, 83
projT1, 86
projT2, 86
Prompt, 143
Proof, 49, 145
Proof editing, 143
Proof General, 316
Proof rendering, 311
Proof term, 143
Proof with, 207
Prop, 31, 96
Proposition, 49
Pwd, 135
Qed, 49, 144
qualid, 76
Qualified identifiers, 69
Quantifiers, 83
Quit, 139
quote, 194, 239
Record, 51
Recursion, 89
Recursive arguments, 114
Recursive Extraction, 363
Recursive Extraction Module, 363
red, 167
refine, 152, 223
refl_equal, 84
refl_identity, 85
reflexivity, 185
Remark, 49, 145
remember, 157
Remove LoadPath, 136
Remove Printing If ident, 57
Remove Printing Let ident, 57
rename, 154
repeat, 214
replace ... with, 185
Require, 134
Require Export, 134
ReservedNotation, 283
Reset, 137
Reset Extraction Inline, 365
Reset Initial, 139
Resource file, 298
Restart, 147
Restore State, 139
Resume, 146
rev, 94
revert, 161
rewrite, 183
rewrite ->, 183
rewrite <-, 184
rewrite ... at, 184
rewrite ... by, 184
rewrite ... in, 184
right, 86, 171
ring, 200, 381, 382
ring_simplify, 200, 382
rtauto, 197
S, 85
Save, 49, 144
Scheme, 208, 225
Scheme Equality, 208
Schemes, 208
Script file, 133
Search, 128
SearchAbout, 129
SearchPattern, 130
SearchRewrite, 131
Section, 61
Sections, 60
Set, 31, 96
set, 157
Set Contextual Implicit, 76
Set Elimination Schemes, 208
Set Equality Scheme, 208
Set Extraction AutoInline, 365
Set Extraction Optimize, 364
Set Firstorder Depth, 198
Set Hyps Limit, 149
Set Implicit Arguments, 75
Set Ltac Debug, 222
Set Maximal Implicit Insertion, 76
Set Printing All, 80
Set Printing Coercion, 343
Set Printing Coercions, 343
Set Printing Depth, 140
Set Printing Implicit, 77
Set Printing Implicit Defensive, 77
Set Printing Matching, 56
Set Printing Notations, 284
Set Printing Synth, 56
Set Printing Universes, 80
Set Printing Width, 140
Set Printing Wildcard, 56
Set Reversible Pattern Implicit, 76
Set Silent, 140
Set Strict Implicit, 75
Set Strongly Strict Implicit, 75
Set Transparent Obligations, 378
Set Undo, 147
Set Virtual Machine, 142
Set Whelp Getter, 132
Set Whelp Server, 132
setoid_reflexivity, 402
setoid_replace, 395, 402
setoid_rewrite, 402
setoid_symmetry, 402
setoid_transitivity, 402
Show, 147
Show Conjectures, 148
Show Existentials, 148
Show Implicits, 147
Show Intro, 148
Show Intros, 148
Show Proof, 148
Show Script, 148
Show Tree, 148
Show XML Proof, 313
sig, 86
sig2, 86
sigT, 86
sigT2, 86
Silent mode, 140
simpl, 167
simpl ... in, 168
simple apply, 157
simple apply ... in, 160
simple destruct, 176
simple eapply ... in, 160
simple induction, 174
simple inversion, 193
simple inversion ... as, 193
simplify_eq, 190
snd, 85
solve, 215
Solve Obligations, 378
Some, 85
sort, 32
Sort-polymorphism of inductive families, 107
Sorts, 31, 96
specialize, 159
specif, 31
split, 171
split_Rabs, 93
split_Rmult, 93
step1, 186
stepr, 186
Strategy, 142
strategy, 30
Structure, 344
SubClass, 343
subgoal, 151
subst, 186
Substitution, 98
sum, 85
sumbool, 86
sumor, 86
Suspend, 146
sym_eq, 84
sym_not_eq, 84
symmetry, 185
symmetry in, 185
tactic, 151
Tactic Definition, 209
Tactic macros, 209
Tacticals, 212
tactic_1, tactic_2, 212
tactic_0, [tactic_1] | ... | tactic_n, 212
abstract, 220
do, 214
fail, 216
first, 215
idtac, 215
info, 220
||, 215
repeat, 214
solve, 215
try, 214
Tactics, 151
tail, 94
tauto, 196
term, 32
Terms, 31
Test Ltac Debug, 222
Test Printing Depth, 140
Test Printing If for ident, 57
Test Printing Let for ident, 57
Test Printing Matching, 56
Test Printing Synth, 56
Test Printing Width, 140
Test Printing Wildcard, 56
Test Virtual Machine, 142
Test Whelp Getter, 132
Test Whelp Server, 132
Theorem, 49, 144
Theories, 81
Time, 140
trans_eq, 84
transitivity, 186
Transparent, 141
trivial, 195
true, 83
try, 214
tt, 85
Type, 31, 96
type, 31
type of
in Ltac, 220
Type of constructor, 105
type_scope, 290
Typeclasses eauto, 354
Typeclasses Opaque, 354
Typeclasses Transparent, 354
Typing rules, 98, 152
App, 99, 158
Ax, 99
Const, 99
Conv, 101, 154, 162
Fix, 114
Lam, 99, 154
Let, 99, 154
match, 112
Prod, 99
Prod (impredicative Set), 117
Var, 99, 152
Undo, 146
Unfocus, 147
unfold, 168
unfold ... in, 168
unit, 85
Unset Contextual Implicit, 76
Unset Extraction AutoInline, 365
Unset Extraction Optimize, 365
Unset Hyps Limit, 149
Unset Implicit Arguments, 75
Unset Ltac Debug, 222
Unset Maximal Implicit Insertion, 76
Unset Printing All, 80
Unset Printing Coercion, 343
Unset Printing Coercions, 343
Unset Printing Depth, 140
Unset Printing Implicit, 77
Unset Printing Implicit Defensive, 77
Unset Printing Matching, 56
Unset Printing Notations, 284
Unset Printing Synth, 56
Unset Printing Universes, 80
Unset Printing Width, 140
Unset Printing Wildcard, 56
Unset Reversible Pattern Implicit, 76
Unset Silent, 140
Unset Strict Implicit, 75
Unset Strongly Strict Implicit, 75
Unset Undo, 147
Unset Virtual Machine, 142

value, 87
Variable, 38
Variable (and coercions), 342
Variables, 38
vm_compute, 166, 167

Well founded induction, 89
Well foundedness, 89
well_founded, 89
Whelp Elim, 133
Whelp Hint, 133
Whelp Instance, 133
Whelp Locate, 132
Whelp Match, 132
Write State, 139

XML exportation, 311

ζ-reduction, 100
Tactics Index

||, 215
;|, 212
;[|...|...], 212

abstract, 220
absurd, 165
admit, 163
apply, 156
apply ... with, 156
apply ... in, 159
assert, 158
assert as, 159
assert by, 159
assumption, 152
auto, 195
autorewrite, 201
case, 176
case_eq, 176
cbv, 166
change, 162
change ... in, 162
classical_left, 195
classical_right, 195
clear, 153
clearbody, 153
cofix, 163
compare, 187
compute, 166
congruence, 198
constructor, 170
contradict, 165
contradiction, 165
cut, 158
cutrewrite, 184
decide equality, 187
decompose, 181
decompose record, 182
decompose sum, 182
dependent destruction, 181
dependent induction, 180
dependent induction ...
generalizing, 181
dependent inversion, 192
dependent inversion ... as, 192
dependent inversion ... as ...
with, 193
dependent inversion ... with, 192
dependent inversion_clear, 192
dependent inversion_clear ...
as, 192
dependent inversion_clear ...
as ... with, 193
dependent inversion_clear ...
with, 193
dependent rewrite ->, 190
dependent rewrite <->, 190
destruct, 174
discriminate, 187
discrR, 93
do, 214
double induction, 179
eapply, 156, 224
eapply ... in, 160
eassumption, 152
eauto, 196
ecase, 176
econstructor, 171
edestruct, 175
ediscriminate, 187
eelim, 174
eexact, 152
eexists, 171
einduction, 173
einjection, 188
Tactics Index

eleleft, 171
elim ... using, 174
elimtype, 174
erewrite, 184
eright, 171
esimplify_eq, 190
esplit, 171
evar, 163
exact, 152
exists, 171
f_equal, 186
fail, 216
field, 200, 388
field_simplify, 200, 388
field_simplify_eq, 200, 388
first, 215
firstorder, 198
firstorder using, 198
firstorder with, 198
firstorder tactic, 198
fix, 162
fold, 169
fourier, 201
functional induction, 182, 227
gappa, 405
generalize, 161
generalize dependent, 161
hnf, 167
idtac, 215
induction, 171
info, 220
injection, 188
injection ... as, 190
instantiate, 163
intro, 154
intro after, 155
intro at bottom, 155
intro at top, 155
intro before, 155
intros, 154
intros intro_pattern, 176
intros until, 155
intuition, 197
inversion, 191, 229
inversion ... as, 191
inversion ... as ... in, 192
inversion ... in, 192
inversion ... using, 193
inversion ... using ... in, 193
inversion_clear, 191
inversion_clear ... as ... in, 192
inversion_clear ... in, 192
inversion_cleardots as, 192
lapply, 157
lazy, 166
left, 171
legacy field, 393
legacy ring, 390
move, 153
omega, 200, 355
pattern, 169
pose, 157
pose proof, 159
progress, 215
quote, 194, 239
red, 167
refine, 152, 223
reflexivity, 185
remember, 157
rename, 154
repeat, 214
replace ... with, 185
revert, 161
rewrite, 183
rewrite ->, 183
rewrite <-, 184
rewrite ... at, 184
rewrite ... by, 184
rewrite ... in, 184
right, 171
ring, 200, 381, 382
ring_simplify, 200, 382
rtauto, 197
set, 157
setoid_replace, 395
simpl, 167
simpl ... in, 168
simple apply, 157
simple apply ... in, 160
simple destruct, 176
simple eapply ... in, 160
simple induction, 174
simple inversion, 193
simple inversion ... as, 193
simplify_eq, 190
solve, 215
specialize, 159
split, 171
split_Rabs, 93
split_Rmult, 93
stepl, 186
stepr, 186
subst, 186
symmetry, 185
symmetry in, 185
tauto, 196
transitivity, 186
trivial, 195
try, 214
unfold, 168
unfold ... in, 168
vm_compute, 166, 167
Vernacular Commands Index

Abort, 145
About, 127
Add Field, 200, 389
Add Legacy Abstract Ring, 392
Add Legacy Abstract Semi Ring, 392
Add Legacy Field, 393
Add Legacy Ring, 390, 392
Add Legacy Semi Ring, 390, 392
Add LoadPath, 136
Add ML Path, 137
Add Morphism, 403
Add Parametric Morphism, 398
Add Parametric Relation, 397
Add Printing If ident, 57
Add Printing Let ident, 57
Add Rec LoadPath, 136
Add Rec ML Path, 137
Add Relation, 397
Add Ring, 200, 384
Add Setoid, 403
Admit Obligations, 378
Admitted, 49, 144
Arguments Scope, 288
Axiom, 38
Axiom (and coercions), 342
Back, 138
BackTo, 139
Backtrack, 138
Bind Scope, 289
Canonical Structure, 78
Cd, 136
Check, 128
Class, 353
Close Scope, 288
Coercion, 80, 341, 342
CoFixpoint, 48
CoFixpoint ... where ..., 284
CoInductive, 45
CoInductive (and coercions), 342
Combined Scheme, 209, 226
Compute, 128
Conjecture, 38
Corollary, 49
Create HintDb, 202
Declare Implicit Tactic, 207
Declare Left Step, 186
Declare ML Module, 135
Declare Right Step, 186
Defined, 49, 144
Definition, 39, 145
Delimit Scope, 288
Derive Dependent Inversion, 193
Derive Dependent Inversion_clear, 193
Derive Inversion, 193
Derive Inversion_clear, 193
Drop, 139
End, 61, 63, 64
Eval, 128
Example, 39
Existential, 146
Existing Instance, 354
Export, 68
Extract Constant, 366
Extract Inductive, 367
Extraction, 128, 363
Extraction Blacklist, 367
Extraction Inline, 365
Extraction Language, 364
Extraction Module, 363
Extraction NoInline, 365
Fact, 49, 145
Program Fixpoint, 377
Program Instance, 350, 354
Program Lemma, 378
Proof, 49, 145
Proof with, 207
Proposition, 49
Pwd, 135
Qed, 49, 144
Quit, 139
Record, 51
Recursive Extraction, 363
Recursive Extraction Module, 363
Remark, 49, 145
Remove LoadPath, 136
Remove Printing If ident, 57
Remove Printing Let ident, 57
Require, 134
Require Export, 134
ReservedNotation, 283
Reset, 137
Reset Extraction Inline, 365
Reset Initial, 139
Restart, 147
Restore State, 139
Resume, 146
Save, 49, 144
Scheme, 208, 225
Scheme Equality, 208
Search, 128
SearchAbout, 129
SearchPattern, 130
SearchRewrite, 131
Section, 61
Set Contextual Implicit, 76
Set Elimination Schemes, 208
Set Equality Scheme, 208
Set Extraction AutoInline, 365
Set Extraction Optimize, 364
Set Firstorder Depth, 198
Set Hyps Limit, 149
Set Implicit Arguments, 75
Set Ltac Debug, 222
Set Maximal Implicit Insertion, 76
Set Printing All, 80
Set Printing Coercion, 343
Set Printing Coercions, 343
Set Printing Depth, 140
Set Printing Implicit, 77
Set Printing Implicit Defensive, 77
Set Printing Matching, 56
Set Printing Notations, 284
Set Printing Synth, 56
Set Printing Universes, 80
Set Printing Width, 140
Set Printing Wildcard, 56
Set Reversible Pattern Implicit, 76
Set Silent, 140
Set Strict Implicit, 75
Set Strongly Strict Implicit, 75
Set Transparent Obligations, 378
Set Undo, 147
Set Virtual Machine, 142
Set Whelp Getter, 132
Set Whelp Server, 132
setoid_reflexivity, 402
setoid_replace, 402
setoid_rewrite, 402
setoid_symmetry, 402
setoid_transitivity, 402
Show, 147
Show Conjectures, 148
Show Existentials, 148
Show Implicit, 147
Show Intro, 148
Show Intros, 148
Show Proof, 148
Show Script, 148
Show Tree, 148
Show XML Proof, 313
Solve Obligations, 378
Strategy, 142
Structure, 344
SubClass, 343
Suspend, 146
Tactic Definition, 209
Test Ltac Debug, 222
Test Printing Depth, 140
Test Printing If for ident, 57
<table>
<thead>
<tr>
<th>Vernacular Commands Index</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test Printing Let for <code>ident</code>, 57</td>
</tr>
<tr>
<td>Test Printing Matching, 56</td>
</tr>
<tr>
<td>Test Printing Synth, 56</td>
</tr>
<tr>
<td>Test Printing Width, 140</td>
</tr>
<tr>
<td>Test Printing Wildcard, 56</td>
</tr>
<tr>
<td>Test Virtual Machine, 142</td>
</tr>
<tr>
<td>Test Whelp Getter, 132</td>
</tr>
<tr>
<td>Test Whelp Server, 132</td>
</tr>
<tr>
<td>Theorem, 49, 144</td>
</tr>
<tr>
<td>Time, 140</td>
</tr>
<tr>
<td>Transparent, 141</td>
</tr>
<tr>
<td>Typeclasses eauto, 354</td>
</tr>
<tr>
<td>Typeclasses Opaque, 354</td>
</tr>
<tr>
<td>Typeclasses Transparent, 354</td>
</tr>
<tr>
<td>Undo, 146</td>
</tr>
<tr>
<td>Unfocus, 147</td>
</tr>
<tr>
<td>Unset Contextual Implicit, 76</td>
</tr>
<tr>
<td>Unset Extraction AutoInline, 365</td>
</tr>
<tr>
<td>Unset Extraction Optimize, 365</td>
</tr>
<tr>
<td>Unset Hyps Limit, 149</td>
</tr>
<tr>
<td>Unset Implicit Arguments, 75</td>
</tr>
<tr>
<td>Unset Ltac Debug, 222</td>
</tr>
<tr>
<td>Unset Maximal Implicit Insertion, 76</td>
</tr>
<tr>
<td>Unset Printing All, 80</td>
</tr>
<tr>
<td>Unset Printing Coercion, 343</td>
</tr>
<tr>
<td>Unset Printing Coercions, 343</td>
</tr>
<tr>
<td>Unset Printing Depth, 140</td>
</tr>
<tr>
<td>Unset Printing Implicit, 77</td>
</tr>
<tr>
<td>Unset Printing Implicit Defensive, 77</td>
</tr>
<tr>
<td>Unset Printing Matching, 56</td>
</tr>
<tr>
<td>Unset Printing Notations, 284</td>
</tr>
<tr>
<td>Unset Printing Synth, 56</td>
</tr>
<tr>
<td>Unset Printing Universes, 80</td>
</tr>
<tr>
<td>Unset Printing Width, 140</td>
</tr>
<tr>
<td>Unset Printing Wildcard, 56</td>
</tr>
<tr>
<td>Unset Reversible Pattern Implicit, 76</td>
</tr>
<tr>
<td>Unset Silent, 140</td>
</tr>
<tr>
<td>Unset Strict Implicit, 75</td>
</tr>
<tr>
<td>Unset Strongly Strict Implicit, 75</td>
</tr>
<tr>
<td>Unset Undo, 147</td>
</tr>
<tr>
<td>Unset Virtual Machine, 142</td>
</tr>
</tbody>
</table>

Variable (and coercions), 342
Variables, 38
Whelp Elim, 133
Whelp Hint, 133
Whelp Instance, 133
Whelp Locate, 132
Whelp Match, 132
Write State, 139
Index of Error Messages

ident₂ not found, 154
ident₁ not found, 153
ident already exists, 38–40, 50, 376
ident not found, 153

A record cannot be recursive, 53
already exists, 145
Argument of match does not evaluate to a term, 217
arguments of ring_simplify do not have all the same type, 383
Attempt to save an incomplete proof, 144

bad lemma for decidability of equality, 386
Bad magic number, 135
bad ring structure, 386
Bound head variable, 203

Can’t find file ident on loadpath, 133
cannot be used as a hint, 203
Cannot build functional inversion principle, 59
Cannot define graph for ident…., 59
Cannot define principle(s) for ident…., 59
cannot find a declared ring structure for equality term, 383
cannot find a declared ring structure over term, 383
Cannot find induction information on qualid, 183
Cannot find inversion information for hypothesis ident, 194
Cannot find library foo in loadpath, 134
Cannot find the source class of qualid, 341
Cannot infer a term for this placeholder, 71, 152
Cannot load qualid: no physical path bound to dirpath, 134
Cannot move ident₁ after ident₂: it occurs in ident₂, 153

Cannot move ident₁ after ident₂: it depends on ident₂, 153
Cannot recognize class₁ as a source class of qualid, 341
Cannot solve the goal, 215
Cannot use mutual definition with well-founded recursion or measure, 59
Compiled library ident.vo makes inconsistent assumptions over library qualid, 134
does not denote an evaluable constant, 168
does not respect the inheritance uniform condition, 341

Error: The term “term” has type “type” while it is expected to have type “type”, 39

Failed to progress, 215
File not found on loadpath : string, 135
Found target class class instead of class₂, 341
Funclass cannot be a source class, 341
goal does not satisfy the expected preconditions, 189
Goal is solvable by congruence but some arguments are missing. Try congruence with . . ., replacing metavariables by arbitrary terms., 199

Hypothesis ident must contain at least one Function, 194

I couldn’t solve goal, 199
I don’t know how to handle dependent equality, 199

Impossible to unify . . . with . . ., 156
Impossible to unify . . . with . . ., 185
In environment . . . the term: term₂ does not have type term₁, 376
invalid argument, 152
is already a coercion, 341
is already used, 154
is not a function, 341
is not a module, 68
is not an inductive type, 203
is used in the conclusion, 153
is used in the hypothesis, 153

Loading of ML object file forbidden in a native Coq, 135

Module/section module not found, 129
must be a transparent constant, 342

name ident is already used, 154
No applicable tactic, 215
No argument name ident, 59
No discriminable equalities, 188
No focused proof, 143, 147
No focused proof (No proof-editing in progress), 145, 146
No focused proof to restart, 147
No matching clauses for match, 217
No matching clauses for match goal, 219
No primitive equality found, 187
No product even after head-reduction, 154, 155
No proof-editing in progress, 146
No such assumption, 152, 165
No such binder, 164
no such entry, 137
No such goal, 147
No such hypothesis, 155, 170
No such hypothesis in current goal, 155
No such label ident, 63
No such proof, 146
Non strictly positive occurrence of ident in type, 42
not a context variable, 220
not a defined object, 127
Not a discriminable equality, 187
Not a primitive equality, 189
Not a projectable equality but a discriminable one, 189
Not a proposition or a type, 158
Not a valid (semi)ring theory, 392
not a valid ring equation, 383
Not an exact proof, 152

Not an inductive product, 170, 172
Not convertible, 162
not declared, 203, 341
Not enough constructors, 170
Not reducible, 167
Not the right number of induction arguments, 183
Not the right number of missing arguments, 156, 164
Nothing to do, it is an equality between convertible terms, 189

omega can’t solve this system, 356
omega: Can’t solve a goal with equality on type, 356
omega: Can’t solve a goal with non-linear products, 356
omega: Can’t solve a goal with proposition variables, 356
omega: Not a quantifier-free goal, 356
omega: Unrecognized atomic proposition: prop, 356
omega: Unrecognized predicate or connective: ident, 356
omega: Unrecognized proposition, 356

Proof is not complete, 220
quote: not a simple fixpoint, 194, 240

Reached begin of command history, 138
ring operation should be declared as a morphism, 386

Signature components for label ident do not match, 63
Sortclass cannot be a source class, 341
Statement without assumptions, 160

Tactic Failure message (level n), 216
Tactic generated a subgoal identical to the original goal, 183
terms do not have convertible types, 185
The conclusion is not a substitutive equation, 185
The conclusion of type is not valid; it must be built from ident, 42
The file ident.vo contains library dirpath and not library dirpath’, 135
The recursive argument must be specified, 59
The reference qualid was not found in the current environment, 128, 129, 141
the term form has type . . . which should be Set, Prop or Type, 143, 144
The term provided does not end with an equation, 183
The numth argument of ident must be ident’ in type, 42
This is not the last opened module, 63
This is not the last opened module type, 64
This is not the last opened section, 61

Unable to apply, 160
Unable to find an instance for the variables ident . . . ident, 156, 172
Undo stack would be exhausted, 146
Universe inconsistency, 96
# List of Figures

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Figure Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>Syntax of terms</td>
<td>34</td>
</tr>
<tr>
<td>1.2</td>
<td>Syntax of terms (continued)</td>
<td>35</td>
</tr>
<tr>
<td>1.3</td>
<td>Syntax of sentences</td>
<td>39</td>
</tr>
<tr>
<td>2.1</td>
<td>Syntax for the definition of Record</td>
<td>53</td>
</tr>
<tr>
<td>2.2</td>
<td>Syntax of Record projections</td>
<td>56</td>
</tr>
<tr>
<td>2.3</td>
<td>Syntax of modules</td>
<td>64</td>
</tr>
<tr>
<td>2.4</td>
<td>Syntax for explicitly giving implicit arguments</td>
<td>78</td>
</tr>
<tr>
<td>3.1</td>
<td>Notations in the initial state</td>
<td>84</td>
</tr>
<tr>
<td>3.2</td>
<td>Syntax of formulas</td>
<td>84</td>
</tr>
<tr>
<td>3.3</td>
<td>Syntax of data-types and specifications</td>
<td>87</td>
</tr>
<tr>
<td>3.4</td>
<td>Definition of the scope for integer arithmetics ((\mathbb{Z}_scope))</td>
<td>93</td>
</tr>
<tr>
<td>3.5</td>
<td>Definition of the scope for natural numbers ((\text{nat}_scope))</td>
<td>94</td>
</tr>
<tr>
<td>3.6</td>
<td>Definition of the scope for real arithmetics ((\mathbb{R}_scope))</td>
<td>94</td>
</tr>
<tr>
<td>3.7</td>
<td>Definition of the scope for lists ((\text{list}_scope))</td>
<td>96</td>
</tr>
<tr>
<td>9.1</td>
<td>Syntax of the tactic language</td>
<td>215</td>
</tr>
<tr>
<td>9.2</td>
<td>Syntax of the tactic language (continued)</td>
<td>216</td>
</tr>
<tr>
<td>9.3</td>
<td>Tactic toplevel definitions</td>
<td>216</td>
</tr>
<tr>
<td>10.1</td>
<td>Definition of the permutation predicate</td>
<td>245</td>
</tr>
<tr>
<td>10.2</td>
<td>Permutation tactic</td>
<td>246</td>
</tr>
<tr>
<td>10.3</td>
<td>Deciding intuitionistic propositions (1)</td>
<td>247</td>
</tr>
<tr>
<td>10.4</td>
<td>Deciding intuitionistic propositions (2)</td>
<td>248</td>
</tr>
<tr>
<td>10.5</td>
<td>Type isomorphism axioms</td>
<td>249</td>
</tr>
<tr>
<td>10.6</td>
<td>Type isomorphism tactic (1)</td>
<td>250</td>
</tr>
<tr>
<td>10.7</td>
<td>Type isomorphism tactic (2)</td>
<td>251</td>
</tr>
<tr>
<td>11.1</td>
<td>Syntax of mathematical proof commands</td>
<td>255</td>
</tr>
<tr>
<td>11.2</td>
<td>Correspondence between basic forward steps and conclusion steps</td>
<td>262</td>
</tr>
<tr>
<td>12.1</td>
<td>Syntax of the variants of Notation</td>
<td>287</td>
</tr>
<tr>
<td>15.1</td>
<td>CoQIDE main screen</td>
<td>320</td>
</tr>
<tr>
<td>15.2</td>
<td>CoQIDE: the query window</td>
<td>321</td>
</tr>
<tr>
<td>17.1</td>
<td>Syntax of classes</td>
<td>342</td>
</tr>
</tbody>
</table>